Categories and Quantum Informatics

Week 5: Adjoint functors

Chris Heunen

THE UNIVERSITY of EDINBURGH
informatics

Overview

- Generalised dual objects: $\frac{\text{Adjoint functors}}{\text{functors}} = \frac{\text{dual objects}}{\text{objects}}$
- Adjoint functors are everywhere: many examples
- Recursion: closed categories

Idea

Think:

$$\begin{aligned} \text{category} &= \text{country} \\ \text{object} &= \text{citizen} \\ \text{morphism} &= \text{speaking in country's language} \\ \text{functor} &= \text{translation} \end{aligned}$$

Equivalence: doesn't matter whether I travel to you and speak your language, or you travel to me and speak my language.

Some things get lost in translation; adjunction is next best thing.

Adjunction: first definition

Let $\mathbf{C} \xrightarrow{F} \mathbf{D}$ and $\mathbf{D} \xrightarrow{G} \mathbf{C}$ be functors. An adjunction between F and G is a natural bijection

$$\mathbf{D}(F(C), D) \simeq \mathbf{C}(C, G(D))$$

Say F is left adjoint to G, and G is right adjoint to F, and write $F \dashv G$.

Free vector space

Forgetful functor **FVect** $\to$ **Set** has left adjoint F

$$F(B) = \{\varphi\colon B \to \mathbb{C} \mid \varphi(b) \neq 0 \text{ for only finitely many } b \in B\}$$

That is, $F(B)$ is free vector space with basis B, consisting of 'formal' linear combinations $\sum_{b \in B} \varphi(b) \cdot b$. E.g. $F(\{1, \ldots, n\}) \simeq \mathbb{C}^n$.

Free monoid

Forgetful functor **Monoid** $\to$ **Set** has left adjoint F

$$F(A) = \{\text{words made up of zero of more letters from } A\}$$

That is, $F(A)$ is free monoid on A. E.g.

$$F(\{\bullet\}) \simeq \mathbb{N} \qquad F(\{0,1\}) = \{\emptyset, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$$

Free category

Forgetful functor **Cat** $\longrightarrow$ **Graph** has left adjoint F

$$F(V,E) = \left[\begin{array}{l}\text{objects: vertices}\\ \text{morphisms: sequences of edges}\end{array}\right]$$

That is, $F(V,E)$ is the free category on (V,E). E.g.

$$F\left(\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \end{tikzcd}\right) = \left(\begin{tikzcd} & \bullet \arrow[dr] \arrow[loop above] & \\ \bullet \arrow[ur] \arrow[rr] \arrow[loop left] & & \bullet \arrow[loop right] \end{tikzcd}\right)$$

Freeness

Let $\mathbf{C}$ be category of structured sets and structure-preserving maps.
There is forgetful functor $U\colon \mathbf{C} \to \mathbf{Set}$. Free functor is left adjoint.

Freeness

Let $\mathbf{C}$ be category of structured sets and structure-preserving maps. There is forgetful functor $U\colon \mathbf{C} \to \mathbf{Set}$. Free functor is left adjoint.

- Often, any object is a quotient of a free one

Freeness

Let $\mathbf{C}$ be category of structured sets and structure-preserving maps. There is forgetful functor $U \colon \mathbf{C} \to \mathbf{Set}$. Free functor is left adjoint.

- Often, any object is a quotient of a free one
- free group: words in letters a or a^{-1},
  identifying aa^{-1} and $a^{-1}a$ with empty word
- free abelian group: furthermore identify ab with ba
  alternatively: functions $A \to \mathbb{Z}$ with finite support

Freeness

Let $\mathbf{C}$ be category of structured sets and structure-preserving maps. There is forgetful functor $U\colon \mathbf{C} \to \mathbf{Set}$. Free functor is left adjoint.

- Often, any object is a quotient of a free one
- free group: words in letters a or a^{-1},
  identifying aa^{-1} and $a^{-1}a$ with empty word
- free abelian group: furthermore identify ab with ba
  alternatively: functions $A \to \mathbb{Z}$ with finite support
- free fields don't exist: not algebraic, as can't divide by zero
- free Hilbert spaces don't exist

Adjoint functors vs adjoints

Write $\mathrm{Sub}(H)$ for set of subspaces of $H \in \mathbf{FHilb}$.
Morphism $f\colon H \to K$ gives monotone function

$$\begin{aligned} \mathrm{Sub}(f)\colon \mathrm{Sub}(H) &\to \mathrm{Sub}(K)^{\mathrm{op}} \\ U &\mapsto f(U)^{\perp} \end{aligned}$$

Adjoint matrices give adjoint functors: $\mathrm{Sub}(f) \dashv \mathrm{Sub}(f^{\dagger})$
Conversely, if $\mathrm{Sub}(f) \dashv \mathrm{Sub}(g)$ then $zf = g^{\dagger}$ for some $z \in \mathbb{C}$.

Quantifiers

Write $p\colon A \times B \to A$ for projection in **Set**.
Write $\mathcal{P}(A)$ for powerset of A, regarded as a category.
Functor

$$\begin{aligned} p^*\colon \mathcal{P}(A) &\to \mathcal{P}(A \times B) \\ U &\mapsto p^{-1}(U) = \{(a,b) \in A \times B \mid a \in V\} \end{aligned}$$

has left adjoint

$$\begin{aligned} \exists\colon \mathcal{P}(A \times B) &\to \mathcal{P}(A) \\ V &\mapsto \{a \in A \mid \exists b \in B\colon (a,b) \in V\} \end{aligned}$$

and right adjoint

$$\begin{aligned} \forall\colon \mathcal{P}(A \times B) &\to \mathcal{P}(A) \\ V &\mapsto \{a \in A \mid \forall b \in B\colon (a,b) \in V\} \end{aligned}$$

Terminal object

Let $\mathbf{C}$ be any category. Write $\mathbf{1}$ for the category with one morphism. There is unique functor $\mathbf{C} \to \mathbf{1}$. Functor $\mathbf{1} \to \mathbf{C}$ is choice of object of $\mathbf{C}$.

Right adjoint functors to $\mathbf{C} \to \mathbf{1}$ are precisely terminal objects in $\mathbf{C}$.

Adjunction: second definition

Let $\mathbf{C} \xrightarrow{F} \mathbf{D}$ and $\mathbf{D} \xrightarrow{G} \mathbf{C}$ be functors.
An adjunction $F \dashv G$ consists of natural transformations

$$\eta_C \colon C \longrightarrow G(F(C)) \qquad\qquad \varepsilon_D \colon F(G(D)) \longrightarrow D$$

such that:

$$\begin{tikzcd}
F(C) \arrow[r, "F(\eta_C)"] \arrow[dr, "\mathrm{id}_{F(C)}"'] & F(G(F(C))) \arrow[d, "\varepsilon_{F(C)}"] \\
 & F(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[dr, "\mathrm{id}_{G(D)}"'] & G(F(G(D))) \arrow[d, "G(\varepsilon_D)"] \\
 & G(D)
\end{tikzcd}$$

Call η the unit of the adjunction, ε the counit.

Adjunction: definitions equivalent

- Let $\alpha_{C,D} \colon \mathbf{D}(F(C), D) \to \mathbf{C}(C, G(D))$ be natural bijection. Set

$$\eta_C = \alpha_{C,F(C)}(\mathrm{id}_{F(C)}), \qquad \varepsilon_D = \alpha^{-1}_{G(D),D}(\mathrm{id}_{G(D)}).$$

  Triangle equations follow directly from naturality of α.

Adjunction: definitions equivalent

- Let $\alpha_{C,D} \colon \mathbf{D}(F(C), D) \to \mathbf{C}(C, G(D))$ be natural bijection. Set

$$\eta_C = \alpha_{C,F(C)}(\mathrm{id}_{F(C)}), \qquad \varepsilon_D = \alpha^{-1}_{G(D),D}(\mathrm{id}_{G(D)}).$$

  Triangle equations follow directly from naturality of α.

- Let $\eta_C \colon C \to G(F(C))$ and $\varepsilon_D \colon F(G(D)) \to D$ be natural transformations satisfying triangle equations. Define

$$\alpha_{C,D}(f) = G(f) \circ \eta_C$$

  Its inverse is $\alpha^{-1}_{C,D}(g) = \varepsilon_D \circ F(f)$, and both are natural.

Freeness, universally

For every set A there is a function $\eta\colon A \longrightarrow F(A)$ such that,
if $f\colon A \longrightarrow C$ is any other function to an object of $\mathbf{C}$,
then there is a unique morphism $\hat{f}\colon F(A) \longrightarrow C$ with $f = \hat{f} \circ \eta$.

$$\begin{tikzcd}
A \arrow[r, "\text{function } \eta"] \arrow[dr, "\text{function } f"'] & F(A) \arrow[d, dashed, "\text{morphism } \hat{f}"] \\
 & C
\end{tikzcd}$$

Freeness, universally

For every set A there is a function $\eta\colon A \to F(A)$ such that,
if $f\colon A \to C$ is any other function to an object of $\mathbf{C}$,
then there is a unique morphism $\hat{f}\colon F(A) \to C$ with $f = \hat{f} \circ \eta$.

$$\begin{tikzcd}
A \arrow[r, "\text{function } \eta"] \arrow[dr, "\text{function } f"'] & F(A) \arrow[d, dashed, "\text{morphism } \hat{f}"] \\
& C
\end{tikzcd}$$

In general, $F(A)$ is *much* larger than A.

Function spaces

Let B be set, consider functor $L\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ given by $L(A) = A \times B$.
Define functor $R\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ by $R(C) = C^B$ set of functions $B \longrightarrow C$.

Then $L \dashv R$.

$$\text{functions } A \longrightarrow C^B \qquad \underset{\overset{\longleftarrow}{\textit{currying}}}{\overset{\simeq}{\underset{\longrightarrow}{\textit{uncurrying}}}} \qquad \text{functions } A \times B \longrightarrow C$$

Heyting algebras

Let $(P, \leq)$ be partially ordered set with greatest lower bounds.

$$p \wedge q \leq r \quad \iff \quad p \leq q \multimap r.$$

Exponentials

A monoidal category $\mathbf{C}$ is closed when functor $-\otimes B\colon \mathbf{C}\to\mathbf{C}$

$$A \mapsto A\otimes B$$
$$f \mapsto f\otimes \mathrm{id}_B$$

has a right adjoint $R_B\colon \mathbf{C}\to\mathbf{C}$ for each object B.
Object $R_B(C)$ is called exponential, written as $B \multimap C$ or C^B.

Equivalently: for all objects B and C there is $B\multimap C$ and natural evaluation transformation $\varepsilon_{X,Y}\colon X\otimes(X\multimap Y)\to Y$, such that any $f\colon X\otimes Z\to Y$ factors through $\varepsilon_{X,Y}$ via $\hat{f}\colon Z\to(X\multimap Y)$.

$$\begin{tikzcd}
X\otimes Z \arrow[d, dashed, "\mathrm{id}_X\otimes\hat{f}"'] \arrow[dr, "f"] & \\
X\otimes(X\multimap Y) \arrow[r, "\varepsilon_{X,Y}"'] & Y
\end{tikzcd}$$

Recursion

Let **C** be closed monoidal category. Think of objects as data types, and morphisms as functions in a programming language.
Closedness: higher order functions.

Recursion

Let **C** be closed monoidal category. Think of objects as data types, and morphisms as functions in a programming language.
Closedness: higher order functions.

May think of specification of recursive function $X \to Y$ as morphism

$$s\colon (X \multimap Y) \to (X \multimap Y)$$

State $f\colon I \to (X \multimap Y)$ satisfying $s \circ f = f$ is called fixed point of spec.
Corresponds to morphism $X \to Y$ that meets the specification.

Recursion

Let **C** be closed monoidal category. Think of objects as data types, and morphisms as functions in a programming language.
Closedness: higher order functions.

May think of specification of recursive function $X \to Y$ as morphism

$$s\colon (X \multimap Y) \to (X \multimap Y)$$

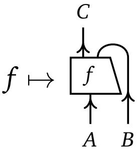
State $f\colon I \to (X \multimap Y)$ satisfying $s \circ f = f$ is called fixed point of spec.
Corresponds to morphism $X \to Y$ that meets the specification.

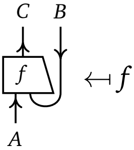
If could pick smallest fixed point systematically,
would have semantics for general recursion.

Compact categories are closed

Define $B \multimap C$ to be $C \otimes B^*$. Natural bijection:

$$\mathbf{C}(A, C \otimes B^*) \simeq \mathbf{C}(A \otimes B, C)$$





Summary

- Generalised dual objects: $\frac{\text{Adjoint functors}}{\text{functors}} = \frac{\text{dual objects}}{\text{objects}}$
- Adjoint functors are everywhere: many examples
- Recursion: closed categories