Categories and Quantum Informatics exercise sheet 4:
Dual objects

Exercise 3.1. Recall the notion of local equivalence from Exercise Sheet 2. In $\text{Hilb}$, we can write a state $C^2 \otimes C^2$ as a column vector

$$\phi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

or as a matrix

$$M_\phi := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) Show that $\phi$ is an entangled state if and only if $M_\phi$ is invertible. (Hint: a matrix is invertible if and only if it has nonzero determinant.)

(b) Show that $M(id_{C^2} \otimes f) \circ \phi = M \phi \circ f^T$, where $C^2 f C^2$ is any linear map and $f^T$ is the transpose of $f$ in the canonical basis of $C^2$.

(c) Use this to show that there are three families of locally equivalent joint states of $C^2 \otimes C^2$.

Exercise 3.2. Pick a basis $\{e_i\}$ for a finite-dimensional vector space $V$, and define $C^\eta V \otimes V$ and $V \otimes V \epsilon C$ by $\eta(1) = \sum_i e_i \otimes e_i$ and $\epsilon(e_i \otimes e_i) = 1$, and $\epsilon(e_i \otimes e_j) = 0$ when $i \neq j$.

(a) Show that this satisfies the snake equations, and hence that $V$ is dual to itself in the category $F\text{Vect}$.

(b) Show that $f^*$ is given by the transpose of the matrix of the morphism $V \overset{f}{\rightarrow} V$ (where the matrix is written with respect to the basis $\{e_i\}$).

(c) Suppose that $\{e_i\}$ and $\{e'_i\}$ are both bases for $V$, giving rise to two units $\eta, \eta'$ and two counits $\epsilon, \epsilon'$. Let $V \overset{f}{\rightarrow} V$ be the ‘change-of-base’ isomorphism $e_i \mapsto e'_i$. Show that $\eta = \eta'$ and $\epsilon = \epsilon'$ if and only if $f$ is (complex) orthogonal, i.e. $f^{-1} = f^*$.

Exercise 3.3. Let $L \dashv R$ in $F\text{Vect}$, with unit $\eta$ and counit $\epsilon$. Pick a basis $\{r_i\}$ for $R$.

(a) Show that there are unique $l_i \in L$ satisfying $\eta(1) = \sum_i r_i \otimes l_i$.

(b) Show that every $l \in L$ can be written as a linear combination of the $l_i$, and hence that the map $R \overset{f}{\rightarrow} L$, defined by $f(r_i) = l_i$, is surjective.

(c) Show that $f$ is an isomorphism, and hence that $\{l_i\}$ must be a basis for $L$.

(d) Conclude that any duality $L \dashv R$ in $F\text{Vect}$ is of the following standard form for a basis $\{l_i\}$ of $L$ and a basis $\{r_i\}$ of $R$:

$$\eta(1) = \sum_i r_i \otimes l_i, \quad \epsilon(l_i \otimes r_j) = \delta_{ij}. \quad (1)$$

Exercise 3.4. Let $L \dashv R$ be dagger dual objects in $F\text{Hilb}$, with unit $\eta$ and counit $\epsilon$.

(a) Use the previous exercise to show that there are an orthonormal basis $\{r_i\}$ of $R$ and a basis $\{l_i\}$ of $L$ such that $\eta(1) = \sum_i r_i \otimes l_i$ and $\epsilon(l_i \otimes r_j) = \delta_{ij}$.
(b) Show that \( \varepsilon(l_i \otimes r_j) = (l_j|l_i) \). Conclude that \( \{l_i\} \) is also an orthonormal basis, and hence that every dagger duality \( L \dashv R \) in \( \mathbf{FHilb} \) has the standard form (1) for orthonormal bases \( \{l_i\} \) of \( L \) and \( \{r_i\} \) of \( R \).

**Exercise 3.5.** Show that any duality \( L \dashv R \) in \( \mathbf{Rel} \) is of the following standard form for an isomorphism \( R \xrightarrow{\eta} L \):

\[
\eta = \{(\bullet,(r,f(r))) \mid r \in R\}, \quad \varepsilon = \{((l,f^{-1}(l)),\bullet) \mid l \in L\}.
\]

Conclude that specifying a duality \( L \dashv R \) in \( \mathbf{Rel} \) is the same as choosing an isomorphism \( R \xrightarrow{\eta} L \), and that dual objects in \( \mathbf{Rel} \) are automatically dagger dual objects.

**Exercise 3.6.** A terminal object is an object 1 such that there is a unique morphism \( A \rightarrow 1 \) for any object \( A \). In a monoidal category with a terminal object, show that: if \( L \dashv R \), then \( R \otimes 1 \simeq 1 \simeq 1 \otimes L \).

**Exercise 3.7.** Show that the trace in \( \mathbf{Rel} \) shows whether a relation has a fixed point.

**Exercise 3.8.** Let \( \mathbf{C} \) be a compact dagger category.

(a) Show that \( \text{Tr}(f) \) is positive when \( A \xrightarrow{f} A \) is a positive morphism.

(b) Show that \( f^* \) is positive when \( A \xrightarrow{A} A \) is a positive morphism.

(c) Show that \( \text{Tr}_{A^*}(f^*) = \text{Tr}_A(f) \) for any morphism \( A \xrightarrow{f} A \).

(d) Show that \( \text{Tr}(g \circ f) \) is positive when \( A \xrightarrow{f} A \) and \( A \xrightarrow{g} A \) are positive morphisms.

**Exercise 3.9.** Show that if \( L \dashv R \) are dagger dual objects, then \( \dim(L)^\dagger = \dim(R) \).