Categories and Quantum Informatics exercise sheet 4: Dual objects

Exercise 3.1. Pick a basis $\{e_i\}$ for a finite-dimensional vector space V, and define $\mathbb{C} \xrightarrow{\eta} V \otimes V$ and $V \otimes V \xrightarrow{\varepsilon} \mathbb{C}$ by $\eta(1) = \sum_i e_i \otimes e_i$ and $\varepsilon(e_i \otimes e_i) = 1$, and $\varepsilon(e_i \otimes e_j) = 0$ when $i \neq j$.

(a) Show that this satisfies the snake equations, and hence that V is dual to itself in the category **FVect**.

(b) Show that f^* is given by the transpose of the matrix of the morphism $V \xrightarrow{f} V$ (where the matrix is written with respect to the basis $\{e_i\}$).

(c) Suppose that $\{e_i\}$ and $\{e'_i\}$ are both bases for V, giving rise to two units η, η' and two counits $\varepsilon, \varepsilon'$. Let $V \xrightarrow{f} V$ be the 'change-of-base' isomorphism $e_i \mapsto e'_i$. Show that $\eta = \eta'$ and $\varepsilon = \varepsilon'$ if and only if f is *(complex) orthogonal*, i.e. $f^{-1} = f^*$.

Exercise 3.2. Let $L \dashv R$ in **FVect**, with unit η and counit ε. Pick a basis $\{r_i\}$ for R.

(a) Show that there are unique $l_i \in L$ satisfying $\eta(1) = \sum_i r_i \otimes l_i$.

(b) Show that every $l \in L$ can be written as a linear combination of the l_i, and hence that the map $R \xrightarrow{f} L$, defined by $f(r_i) = l_i$, is surjective.

(c) Show that f is an isomorphism, and hence that $\{l_i\}$ must be a basis for L.

(d) Conclude that any duality $L \dashv R$ in **FVect** is of the following *standard form* for a basis $\{l_i\}$ of L and a basis $\{r_i\}$ of R:

$$\eta(1) = \sum_i r_i \otimes l_i, \qquad \varepsilon(l_i \otimes r_j) = \delta_{ij}. \tag{1}$$

Exercise 3.3. Let $L \dashv R$ be dagger dual objects in **FHilb**, with unit η and counit ε.

(a) Use the previous exercise to show that there are an orthonormal basis $\{r_i\}$ of R and a basis $\{l_i\}$ of L such that $\eta(1) = \sum_i r_i \otimes l_i$ and $\varepsilon(l_i \otimes r_j) = \delta_{ij}$.

(b) Show that $\varepsilon(l_i \otimes r_j) = \langle l_j | l_i \rangle$. Conclude that $\{l_i\}$ is also an orthonormal basis, and hence that every dagger duality $L \dashv R$ in **FHilb** has the standard form (1) for *orthonormal* bases $\{l_i\}$ of L and $\{r_i\}$ of R.

Exercise 3.4. Show that any duality $L \dashv R$ in **Rel** is of the following *standard form* for an isomorphism $R \xrightarrow{f} L$:

$$\eta = \{(\bullet, (r, f(r))) \mid r \in R\}, \qquad \varepsilon = \{((l, f^{-1}(l)), \bullet) \mid l \in L\}.$$

Conclude that specifying a duality $L \dashv R$ in **Rel** is the same as choosing an isomorphism $R \to L$, and that dual objects in **Rel** are automatically dagger dual objects.

Exercise 3.5. A *terminal object* is an object 1 such that there is a unique morphism $A \to 1$ for any object A. In a monoidal category with a terminal object, show that: if $L \dashv R$, then $R \otimes 1 \simeq 1 \simeq 1 \otimes L$.

Exercise 3.6. Show that the trace in **Rel** shows whether a relation has a fixed point.

Exercise 3.7. Let **C** be a compact dagger category.

(a) Show that $\mathrm{Tr}(f)$ is positive when $A \xrightarrow{f} A$ is a positive morphism.

(b) Show that f^* is positive when $A \xrightarrow{f} A$ is a positive morphism.

(c) Show that $\mathrm{Tr}_{A^*}(f^*) = \mathrm{Tr}_A(f)$ for any morphism $A \xrightarrow{f} A$.

(d) Show that $\mathrm{Tr}(g \circ f)$ is positive when $A \xrightarrow{f,g} A$ are positive morphisms.

Exercise 3.8. Show that if $L \dashv R$ are dagger dual objects, then $\dim(L)^\dagger = \dim(R)$.