Categories and Quantum Informatics exercise sheet 4: Dual objects

Exercise 3.1. (a) Evaluating the snake equation on each e_k gives

$$\begin{aligned}(\mathrm{id}_V \otimes \varepsilon) \circ (\eta \otimes \mathrm{id}_V)(e_k) &= (\mathrm{id}_V \otimes \varepsilon)(\sum_i e_i \otimes e_i \otimes e_k) \\ &= \sum_i e_i \otimes \varepsilon(e_i \otimes e_k) \\ &= e_k,\end{aligned}$$

so indeed $(\mathrm{id}_V \otimes \varepsilon) \circ (\eta \otimes \mathrm{id}_V) = \mathrm{id}_V$; the other snake equation is verified similarly.

(b) Let $(f_{i,j})$ be the matrix of f. So $e_i \xrightarrow{f} \sum_j f_{j,i}e_j$ and $e_i \xrightarrow{f^T} \sum_j f_{i,j}e_j$.
When we evaluate on each e_k we get

$$\begin{aligned}f^*(e_k) &= (\mathrm{id}_V \otimes \varepsilon) \circ (\mathrm{id}_V \otimes f \otimes \mathrm{id}_V) \circ (\eta \otimes \mathrm{id}_V)(e_k) \\ &= (\mathrm{id}_V \otimes \varepsilon) \circ (\mathrm{id}_V \otimes f \otimes \mathrm{id}_V)(\sum_i e_i \otimes e_i \otimes e_k) \\ &= \sum_i (\mathrm{id}_V \otimes \varepsilon)(e_i \otimes f(e_i) \otimes e_k) \\ &= \sum_{ij} e_i \otimes f_{ij}\varepsilon(e_j \otimes e_k) \\ &= \sum_i f_{ik}e_i \\ &= f^{\mathrm{T}}(e_k),\end{aligned}$$

and so $f^* = f^{\mathrm{T}}$.

(c) By Lemma 3.5 we may focus on $\eta = \eta'$ and forget about $\varepsilon = \varepsilon'$. Because $e'_i = \sum_j f_{ij}e_j$, we get

$$\eta'(1) = \sum_i e'_i \otimes e'_i = \sum_{i,j,k} f_{ij}f_{ik}e_j \otimes e_k.$$

This equals $\eta(1) = \sum_i e_i \otimes e_i$ precisely when $\sum_i f_{ij}f_{ik} = \delta_{jk}$ for all j, k. But this happens precisely when $f^{\mathrm{T}} \circ f = \mathrm{id}_V$, since

$$f^T \circ f = \begin{pmatrix} f_{1,1} & \cdots & f_{n,1} \\ \vdots & \ddots & \vdots \\ f_{1,n} & \cdots & f_{n,n} \end{pmatrix} \begin{pmatrix} f_{1,1} & \cdots & f_{1,n} \\ \vdots & \ddots & \vdots \\ f_{n,1} & \cdots & f_{n,n} \end{pmatrix} = \begin{pmatrix} \sum_i f_{i,1}f_{1,i} & \cdots & \sum_i f_{i,1}f_{i,n} \\ \vdots & \ddots & \vdots \\ \sum_i f_{i,n}f_{i,1} & \cdots & \sum_i f_{i,n}f_{i,n} \end{pmatrix}$$

Because f is invertible, this means $f^{\mathrm{T}} = f^{-1}$.

Exercise 3.2. Like any vector in $R \otimes L$, we can write $\eta(1)$ as $\sum_{j=1}^m z_j x_j \otimes y_j$ for $z_j \in \mathbb{C}$, $x_j \in R$, and $y_j \in L$, where m is some finite number. Developing each x_j on the basis $\{r_i\}$ and using bilinearity of the tensor product, we see that we can also write it as $\sum_{i=1}^{n} r_i \otimes l_i$ for $n = \dim(V)$ and $l_i \in L$. If we could also write it as $\sum_{i=1}^{n} r_i \otimes l_i'$, then we would have $0 = \sum_{i=1}^{n} r_i \otimes (l_i - l_i')$. Because r_i forms a basis, it would follow that $l_i = l_i'$ for each i. Hence the l_i are unique.

(a) Use the snake equation:

$$\begin{aligned} l &= \mathrm{id}_L(l) \\ &= (\varepsilon \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \eta)(l) \\ &= (\varepsilon \otimes \mathrm{id}_L)(\sum_i l \otimes r_i \otimes l_i) \\ &= \sum_i \varepsilon(l \otimes r_i) l_i. \end{aligned}$$

(b) Similarly, it follows from the snake equation that $r_i = \sum_k \varepsilon(l_k \otimes r_i) r_k$. Suppose that $l_i = l_j$. Because $\{r_k\}$ are linearly independent, then $\varepsilon(l_i \otimes r_i) = 1$, and $\varepsilon(l_k \otimes r_i) = 0$ for $k \neq i$. Hence $\varepsilon(l_j \otimes r_i) = 1$, and it follows that $i = j$, and so $r_i = r_j$. So f is injective.

(c) First notice that the standard form unit and counit indeed satisfy the snake equation. For the converse, combine the previous parts with **??**.

Exercise 3.3. (a) A Hilbert space is in particular a vector space. In the previous exercise, we may start by choosing $\{r_i\}$ to be orthonormal.

(b) First, compute that $\eta^\dagger(r_i \otimes l_j) = \langle l_i | l_j \rangle$:

$$\begin{aligned} \langle \eta(1) | r_i \otimes l_j \rangle &= \sum_k \langle r_k | r_i \rangle \langle l_k | l_j \rangle \\ &= \langle l_i | l_j \rangle \\ &= \langle 1 | \eta^\dagger (r_i \otimes l_j) \rangle. \end{aligned}$$

Hence dagger duality shows that $\varepsilon(l_i \otimes r_j) = \eta^\dagger \circ \sigma(l_i \otimes r_j) = \eta^\dagger(r_j \otimes l_i) = \langle l_j | l_i \rangle$. But part (a) shows that also $\varepsilon(l_i \otimes r_j) = \delta_{ij}$. Hence $\langle l_i | l_j \rangle = \delta_{ij}$, making $\{l_i\}$ orthonormal.

Exercise 3.4. First notice that the standard form indeed satisfies the snake equations.
Second, if η and ε witness $L \dashv R$, then for each $r \in R$ there exists $l \in L$ such that $(\bullet, (r, l)) \in \eta$ by one snake equation. But there can be at most one such l because of the other snake equation. Thus $f(r) = l$ defines an isomorphism $R \xrightarrow{f} L$ that makes η of the standard form. By **??**, also ε must be of the standard form.
Third, observe that if $f \neq f'$, then $\eta \neq \eta'$. Hence different choices of isomorphism $R \simeq L$ yield different (co)unit maps.
Finally, notice that any isomorphism is a unitary.

Exercise 3.5. We will prove that $L \otimes 0$ is the initial object; that is: for every object X, there exists a unique morphism $L \otimes 0 \to Z$. The isomorphism $L \otimes 0 \cong 0$ follows the from uniqueness of the initial object. The isomorphism $0 \dashv 0 \otimes R$ is done analogously.

Exercise 3.6. Let $X \xrightarrow{R} X$. Compute:

$$\begin{aligned} \mathrm{Tr}(R) &= \varepsilon \circ (R \otimes \mathrm{id}_X) \circ \sigma_{X,X} \circ \eta \\ &= \{((x,x),\bullet) \mid x \in X\} \circ (R \otimes \mathrm{id}_X) \circ \{((x,y),(y,x)) \mid x, y \in X\} \circ \{(\bullet,(x,x)) \mid x \in X\} \\ &= \{((x,x),\bullet) \mid x \in X\} \circ (R \otimes \mathrm{id}_X) \circ \{(\bullet,(x,x)) \mid x \in X\} \\ &= \{((x,x),\bullet) \mid x \in X\} \circ \{(\bullet,(y,x)) \mid (x,y) \in R\} \\ &= \{(\bullet,\bullet) \mid \exists x \in X \colon xRx\}. \end{aligned}$$

So $\mathrm{Tr}(R) = 1$ when R has a fixed point, and $\mathrm{Tr}(R) = 0$ otherwise.

Exercise 3.7. (a) Say $f = g^\dagger \circ g$ for $A \xrightarrow{g} B$. Now use dagger duality:

$$\begin{aligned}
\mathrm{Tr}_A(f) &= \varepsilon_A \circ (g^\dagger \otimes \mathrm{id}_{A^*}) \circ (g \otimes \mathrm{id}_{A^*}) \circ \sigma_{A^*,A} \circ \eta_A \\
&= \varepsilon_A \circ (g^\dagger \otimes \mathrm{id}_{A^*}) \circ \sigma_{A^*,B} \circ (\mathrm{id}_{A^*} \otimes g) \circ \eta_A \\
&= \eta_A^\dagger \circ \sigma_{A,A^*} \circ (g^\dagger \otimes \mathrm{id}_{A^*}) \circ \sigma_{A^*,B} \circ (\mathrm{id}_{A^*} \otimes g) \circ \eta_A \\
&= \eta_A^\dagger \circ (\mathrm{id}_{A^*} \otimes g^\dagger) \circ (\mathrm{id}_{A^*} \otimes g) \circ \eta_A.
\end{aligned}$$

(b) If $f = g^\dagger \circ g$, then $f^* = g^* \circ (g^\dagger)^* = (g^*) \circ (g^*)^\dagger$.

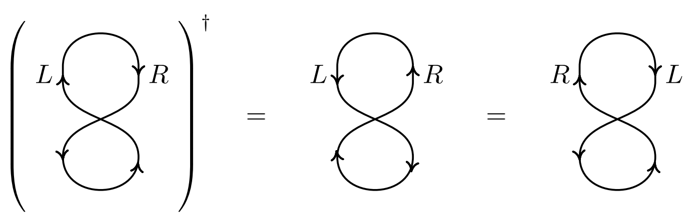
(c)

$$\begin{aligned}
\mathrm{Tr}_{A^*}(f^*) &= \varepsilon_{A^*} \circ (f^* \otimes \mathrm{id}_A) \circ \sigma_{A,A^*} \circ \eta_{A^*} \\
&= \varepsilon_{A^*} \circ (\mathrm{id}_{A^*} \otimes f) \circ \sigma_{A,A^*} \circ \eta_{A^*} \\
&= \varepsilon_A \circ \sigma_{A,A^*} \circ (\mathrm{id}_{A^*} \otimes f) \otimes \sigma_{A,A^*} \circ \sigma_{A^*,A} \circ \eta_A \\
&= \mathrm{Tr}_A(f).
\end{aligned}$$

(d) This is graphically immediately clear.

(e) Suppose $f = a^\dagger \circ a$ and $g = b^\dagger \circ b$; use the cyclic property to see $\mathrm{Tr}(g \circ f) = \mathrm{Tr}((b^\dagger \circ a)^\dagger \circ (b^\dagger \circ a))$, and then use part (a) to see that this scalar is positive.

Exercise 3.8. Graphically:

$\dim(L)^\dagger =$  $= \dim(R)$.