Categories and Quantum Informatics
Week 4: Dual objects

Chris Heunen
Dual objects

Idea:

- Quantum mechanically: maximally entangled states
- Graphically: bending wires
Dual objects

Idea:

- Quantum mechanically: maximally entangled states
- Graphically: bending wires

An object $L$ is left-dual to an object $R$, and $R$ is right-dual to $L$, written $L \rightarrow^\perp R$, when there is a unit morphism $I \xrightarrow{\eta} R \otimes L$ and a counit morphism $L \otimes R \xrightarrow{\varepsilon} I$ such that:

$$
\begin{align*}
L & \xrightarrow{\rho_L^{-1}} L \otimes I \xrightarrow{id_L \otimes \eta} L \otimes (R \otimes L) \\
& \downarrow id_L \quad \downarrow \alpha_{L,R,L}^{-1} \\
L & \quad L \otimes I \quad (L \otimes R) \otimes L \\
& \xleftarrow{\lambda_L} I \otimes L \xleftarrow{\varepsilon \otimes id_L} (L \otimes R) \otimes L \\
& \xleftarrow{\lambda_R^{-1}} I \otimes R \xleftarrow{\eta \otimes id_R} (R \otimes L) \otimes R \\
& \downarrow id_R \quad \downarrow \alpha_{R,L,R}^{-1} \\
R & \xleftarrow{\rho_R} R \otimes I \xleftarrow{id_R \otimes \varepsilon} R \otimes (L \otimes R)
\end{align*}
$$
Snake equations

Draw an object $L$ as a wire with an upward-pointing arrow, and a right dual $R$ as a wire with a downward-pointing arrow.

\[
\begin{array}{c}
\quad \\
\downarrow \\
L \\
\quad \\
\end{array} 
\quad 
\begin{array}{c}
\quad \\
\downarrow \\
R \\
\quad \\
\end{array}
\]

Duality equations become:

\[
\begin{array}{c}
\eta \\
\quad \\
\end{array} 
\quad 
\begin{array}{c}
\quad \\
\varepsilon \\
\quad \\
\end{array}
\]

Also called the snake equations.
Snake equations

Draw an object $L$ as a wire with an upward-pointing arrow, and a right dual $R$ as a wire with a downward-pointing arrow.

The unit $I \xrightarrow{\eta} R \otimes L$ and counit $L \otimes R \xrightarrow{\epsilon} I$ are drawn as bent wires:

Duality equations become:

Also called the snake equations.
Dual Hilbert spaces

**FHilb** has all duals: any finite-dimensional Hilbert space $H$ is both right and left dual to its dual Hilbert space $H^*$, in a canonical way.
Dual Hilbert spaces

\textbf{FHilb} has all duals: any finite-dimensional Hilbert space \( H \) is both right and left dual to its dual Hilbert space \( H^* \), in a canonical way.

The counit \( H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C} \) is:

\[
|\phi\rangle \otimes \langle \psi | \mapsto \langle \psi | \phi \rangle
\]

The unit \( \mathbb{C} \xrightarrow{\eta} H^* \otimes H \) is defined like so, for any orthonormal basis \( |i\rangle \):

\[
1 \mapsto \sum_i \langle i | \otimes | i \rangle
\]
Dual Hilbert spaces

\textbf{FHilb} has all duals: any finite-dimensional Hilbert space $H$ is both right and left dual to its dual Hilbert space $H^*$, in a canonical way.

The counit $H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C}$ is:

$$|\phi\rangle \otimes \langle\psi| \mapsto \langle\psi|\phi\rangle$$

The unit $\mathbb{C} \xrightarrow{\eta} H^* \otimes H$ is defined like so, for any orthonormal basis $|i\rangle$:

$$1 \rightarrow \sum_i \langle i| \otimes |i\rangle$$

Is $\eta$ basis-dependent, but $\varepsilon$ not? No. (Will prove shortly.)

Infinite-dimensional spaces do not have duals. (Will prove later.)
Dual matrices

In $\text{Mat}_\mathbb{C}$, every object $n$ is its own dual, with a canonical choice of $\eta$ and $\varepsilon$ given as follows:

$$\eta : 1 \mapsto \sum_i |i\rangle \otimes |i\rangle$$
$$\varepsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{ij} 1$$
Dual relations

In **Rel**, every object is its own dual, even infinite sets. Unit $1 \xrightarrow{\eta} S \times S$ and counit $S \times S \xrightarrow{\varepsilon} 1$ are:

- $\sim_\eta (s, s)$ for all $s \in S$
- $(s, s) \sim_\varepsilon \bullet$ for all $s \in S$
Names and conames

\textbf{Set} only has duals for singleton sets.

Let \( A \xrightarrow{f} B \) be a morphism in a monoidal category with dualities \( A \dashv A^* \) and \( B \dashv B^* \). Its \textit{name} \( I \xrightarrow{f} A^* \otimes B \) and \textit{coname} \( A \otimes B^* \xleftarrow{f} I \) are:
Names and conames

**Set** only has duals for singleton sets. Let \( A \xrightarrow{f} B \) be a morphism in a monoidal category with dualities \( A \dashv A^* \) and \( B \dashv B^* \). Its name \( I \xrightarrow{\text{name}} A^* \otimes B \) and coname \( A \otimes B^* \xrightarrow{\text{coname}} I \) are:

\[
\begin{align*}
A^* & \quad B \\
\downarrow & \quad \downarrow \\
& \quad f \\
\uparrow & \quad \uparrow \\
A & \quad B \\
\end{align*}
\]

\[
\begin{align*}
I & \quad A^* \otimes B \\
\downarrow & \quad \downarrow \\
& \quad f \\
\uparrow & \quad \uparrow \\
& \quad I \\
\end{align*}
\]

Morphisms can be recovered from their names or conames:

\[
\begin{align*}
B & \quad A \\
\downarrow & \quad \downarrow \\
\quad f \\
\uparrow & \quad \uparrow \\
A & \quad B \\
\end{align*}
\]

\[
\begin{align*}
B & \quad A \\
\downarrow & \quad \downarrow \\
\quad f \\
\uparrow & \quad \uparrow \\
A & \quad B \\
\end{align*}
\]
Names and conames

**Set** only has duals for singleton sets. Let $A \xrightarrow{f} B$ be a morphism in a monoidal category with dualities $A \dashv A^*$ and $B \dashv B^*$. Its name $I \xrightarrow{\lfloor f \rfloor} A^* \otimes B$ and coname $A \otimes B^* \xleftarrow{\lceil f \rceil} I$ are:

![Diagram](image)

Morphisms can be recovered from their names or conames:

![Diagram](image)

In **Set** $I$ is terminal, and so all conames $A \otimes B^* \xleftarrow{\lceil f \rceil} I$ must be equal. If **Set** had duals this would imply all functions $A \rightarrow B$ were equal.
Duals are unique up to iso

In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$. 

Proof: If $L \dashv R$ and $L \dashv R'$, define maps $R \rightarrow R'$ and $R' \rightarrow R$ by:

The snake equations imply that these are inverse. Conversely, if $L \dashv R$ and $R \rightarrow R'$ is invertible, we can construct a duality $L \dashv R'$:

An iso $L \simeq L'$ lets us produce duality $L' \dashv R'$ in a similar way.
Duals are unique up to iso

In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$.

Proof: If $L \dashv R$ and $L \dashv R'$, define maps $R \rightarrow R'$ and $R' \rightarrow R$ by:

$$
\begin{array}{ccc}
R' & \xrightarrow{L} & R \\
\downarrow & & \downarrow \\
R & \xrightarrow{L} & R'
\end{array}
$$

The snake equations imply that these are inverse.
**Duals are unique up to iso**

In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$.

**Proof:** If $L \dashv R$ and $L \dashv R'$, define maps $R \rightarrow R'$ and $R' \rightarrow R$ by:

![Diagram 1](image1.png)

The snake equations imply that these are inverse. Conversely, if $L \dashv R$ and $R \xrightarrow{f} R'$ is invertible, we can construct a duality $L \dashv R'$:

![Diagram 2](image2.png)

An iso $L \simeq L'$ lets us produce duality $L' \dashv R$ in a similar way.
Unit determines counit

If \((L, R, \eta, \varepsilon)\) and \((L, R, \eta, \varepsilon')\) both exhibit duality, then \(\varepsilon = \varepsilon'\).
If \((L, R, \eta, \varepsilon)\) and \((L, R, \eta', \varepsilon)\) both exhibit duality, then \(\eta = \eta'\).
Unit determines counit

If \((L, R, \eta, \varepsilon)\) and \((L, R, \eta, \varepsilon')\) both exhibit duality, then \(\varepsilon = \varepsilon'\).

If \((L, R, \eta, \varepsilon)\) and \((L, R, \eta', \varepsilon)\) both exhibit duality, then \(\eta = \eta'\).

Proof:

\[
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon' \\
\downarrow \\
\varepsilon
\end{array}
= 
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon' \\
\downarrow \\
\varepsilon
\end{array}
\overset{\text{iso}}{=} 
\begin{array}{c}
\varepsilon' \\
\downarrow \\
\varepsilon \\
\downarrow \\
\varepsilon'
\end{array}
= 
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon' \\
\downarrow \\
\varepsilon'
\end{array}
\]
Duals respect tensors

In a monoidal category, $I \vdash I$, and $L \otimes L' \vdash R \otimes R'$ if $L \vdash R$ and $L' \vdash R'$. 
Duals respect tensors

In a monoidal category, \( I \dashv I \), and \( L \otimes L' \dashv R \otimes R' \) if \( L \dashv R \) and \( L' \dashv R' \).

Proof: Taking \( \eta = \lambda_I^{-1} : I \to I \otimes I \) and \( \varepsilon = \lambda_I : I \otimes I \to I \) shows that \( I \dashv I \). Snake equations follow from the coherence theorem.
Duals respect tensors

In a monoidal category, $I \vdash I$, and $L \otimes L' \vdash R \otimes R'$ if $L \vdash R$ and $L' \vdash R'$.

Proof: Taking $\eta = \lambda_I^{-1}: I \rightarrow I \otimes I$ and $\varepsilon = \lambda_I: I \otimes I \rightarrow I$ shows that $I \vdash I$. Snake equations follow from the coherence theorem.

Now suppose $L \vdash R$ and $L' \vdash R'$. We make the new unit and counit maps from the old ones, and compute as follows:
Duals respect braiding

In a braided monoidal category, \( L \dashv R \Rightarrow R \dashv L \).
Duals respect braiding

In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$.

Construct a new duality as follows:

\[ I \xrightarrow{\eta'} L \otimes R \]

\[ R \otimes L \xrightarrow{\varepsilon'} I \]
Duals respect braiding

In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$.

Construct a new duality as follows:

\[
\begin{align*}
I & \xrightarrow{\eta'} L \otimes R \\
R \otimes L & \xrightarrow{\varepsilon'} I
\end{align*}
\]

Test the snake equations:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.4\textwidth]{snake_equation1}}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.4\textwidth]{snake_equation2}}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.4\textwidth]{snake_equation3}}
\end{array}
\end{array}
\end{align*}
\]
For a morphism $A \xrightarrow{f} B$ and chosen dualities $A \dashv A^*$, $B \dashv B^*$, the right dual $B^* \xrightarrow{f^*} A^*$ is defined in the following way:

Represent this graphically by rotating the box for $f$. 

\[
\begin{array}{c}
A^* \\
\downarrow \\
B^* \\
\hline \\
\end{array} 
\begin{array}{c}
A^* \\
\downarrow \\
B^* \\
\hline \\
\end{array} 
\begin{array}{c}
A^* \\
\downarrow \\
B^* \\
\hline \\
\end{array} 
\begin{array}{c}
A^* \\
\downarrow \\
B^* \\
\hline \\
\end{array}
\]
For all morphisms $A \xrightarrow{f} B$ in a monoidal category with chosen duals $A \dashv A^*$ and $B \dashv B^*$:
Duals are functorial

If a monoidal category has chosen right duals, \((-\))^* is a functor.

Proof: Let \(A \xrightarrow{f} B\) and \(B \xrightarrow{g} C\).

\[(g \circ f)^* = f^* \circ g^* = (f \circ g)^*\]

Similarly, \((\text{id}_A)^* = \text{id}_A^*\) follows from the snake equations.
Examples

- In $\textbf{FVect}$ and $\textbf{FHilb}$, right dual of $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{e} \mathbb{C}$ is an arbitrary element of $W^*$.

- In $\textbf{Mat}_\mathbb{C}$, the dual of a matrix is its transpose.

- In $\textbf{Rel}$, the dual of a relation is its converse. So the right duals functor and the dagger functor have the same action: $R^* = R^\dagger$ for all relations $R$. 
Double duals

In monoidal category with chosen right duals, $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$. 
Double duals

In monoidal category with chosen right duals, $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$.

Proof:

$\varepsilon_{A \otimes B}$

$\eta_{(A \otimes B)^*}$
Teleportation

In a monoidal category with right duals, a teleportation procedure is a finite family of effects $e_i : A \otimes A^* \to I$ and unitaries $U_i : A \to A$ with:

\[
e_i U_i = A = A
\]
Teleportation

In a monoidal category with right duals, a teleportation procedure is a finite family of effects $e_i : A \otimes A^* \rightarrow I$ and unitaries $U_i : A \rightarrow A$ with:

\[
\begin{array}{c}
A \\
\downarrow \\
e_i \\
\downarrow \\
A \\
\end{array}
\quad = 
\quad 
\begin{array}{c}
A \\
\downarrow \\
U_i \\
\downarrow \\
A \\
\end{array}
\]

This can be solved to give

\[
\begin{array}{c}
e_i \\
\downarrow \\
A \\
\end{array}
\quad = 
\quad 
\begin{array}{c}
A \\
\downarrow \\
U_i \\
\downarrow \\
A \\
\end{array}
\]
Teleportation

Simplify the history:

\[
\begin{align*}
L & \quad \quad \quad \quad \quad \quad U_i \\
L & \quad \quad \quad \quad \quad \quad U_i \\
\end{align*}
\]

So if the original history occurs, the result is for the state of the original system to be transmitted faithfully. If \( \{ e_i \} \) is a complete set of effects, this will always succeed.
Teleportation

Simplify the history:

So if the original history occurs, the result is for the state of the original system to be transmitted faithfully.

If $\{e_i\}$ is a complete set of effects, this will always succeed.
Teleportation

Simplify the history:

\[ L \xrightarrow{U_i} L \xrightarrow{U_i} L \]

So if the original history occurs, the result is for the state of the original system to be transmitted faithfully.

If \( \{e_i\} \) is a complete set of effects, this will always succeed.
Teleportation

Simplify the history:

So if the original history occurs, the result is for the state of the original system to be transmitted faithfully.
Teleportation

Simplify the history:

So if the original history occurs, the result is for the state of the original system to be transmitted faithfully.

If \( \{e_i\} \) is a complete set of effects, this will always succeed.
Teleportation in $\text{Hilb}$

Choose $L = R = \mathbb{C}^2$ and $\eta^\dagger = \varepsilon = (1 \ 0 \ 0 \ 1)$, and unitaries $U_i$:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

This gives rise to the following family of effects:

\[
\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}
\]

This is a complete set of effects, since it forms a basis for the vector space $\text{Hilb}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C})$. So it is guaranteed to be successful.
Teleportation in Hilb

Choose $L = R = \mathbb{C}^2$ and $\eta^\dagger = \varepsilon = (1 \ 0 \ 0 \ 1)$, and unitaries $U_i$:

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \quad \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

This gives rise to the following family of effects:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix}
$$

This is a complete set of effects, since it forms a basis for the vector space $\text{Hilb}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C})$. So it is guaranteed to be successful.

This is traditional qubit teleportation.
Teleportation in Rel

Choose \( L = R = \{0, 1\} \) and \( \eta^\dagger = \varepsilon = (1 \ 0 \ 0 \ 1) \), and unitaries:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

This gives rise to the following family of effects:

\[
(1 \ 0 \ 0 \ 1) \quad (0 \ 1 \ 1 \ 0)
\]

These form a complete set of effects.
Teleportation in \textbf{Rel}

Choose $L = R = \{0, 1\}$ and $\eta^\dagger = \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, and unitaries:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

This gives rise to the following family of effects:

\[
\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}
\]

These form a complete set of effects.

This is classical encrypted communication with a one-time pad.
Graphical calculus for compact categories

A compact category is symmetric monoidal with chosen duals.
Graphical calculus for compact categories

A compact category is symmetric monoidal with chosen duals.

A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to four-dimensional oriented isotopy.
Graphical calculus for compact categories

A compact category is symmetric monoidal with chosen duals.

A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to four-dimensional oriented isotopy.

Wires of our diagram have arrows, isotopy must preserve them:
Graphical calculus for compact categories

A compact category is symmetric monoidal with chosen duals.

A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to four-dimensional \textit{oriented} isotopy.

Wires of our diagram have arrows, isotopy must preserve them:
Graphical calculus for compact categories

A compact category is symmetric monoidal with chosen duals.

A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to four-dimensional oriented isotopy.

Wires of our diagram have arrows, isotopy must preserve them:
Graphical calculus for compact categories

A compact category is symmetric monoidal with chosen duals.

A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to four-dimensional oriented isotopy.

Wires of our diagram have arrows, isotopy must preserve them:
Intermezzo: ribbon categories

Could have got by with less than symmetric monoidal with duals. Useful in topological quantum computation.

Make some ribbons by cutting long, thin strips from piece of paper. Verify:

\[
\begin{align*}
\text{\includegraphics{ribbon.png}} \quad &= \quad \text{\includegraphics{ribbon.png}}
\end{align*}
\]
Compact dagger categories

In a monoidal dagger category, \( L \rightarrowdash R \iff R \rightarrowdash L \).

Proof: follows directly from axiom \((f \otimes g) \dagger = f^\dagger \otimes g^\dagger\).
Compact dagger categories

In a monoidal dagger category, \( L \dashv R \iff R \dashv L \).
Proof: follows directly from axiom \((f \otimes g)^\dagger = f^\dagger \otimes g^\dagger\).

In a monoidal dagger category, a **dagger dual** is a duality \( A \dashv A^* \)
witnessed by morphisms \( I \xrightarrow{\eta} A^* \otimes A \) and \( A \otimes A^* \xrightarrow{\varepsilon} I \)
satisfying:

\[ \eta \quad = \quad \varepsilon \]
Maximally entangled states

In a compact dagger category, a maximally entangled state is a bipartite state with:

\[
\begin{align*}
\eta & \quad = \\
\eta & \quad = \\
\end{align*}
\]

Proof:

\[
\begin{align*}
\eta & \quad \eta \\
\eta & \quad \eta \\
\end{align*}
\]

\[
\begin{align*}
\eta & \quad = \\
\eta & \quad = \\
\end{align*}
\]
Maximally entangled states

In a compact dagger category, a maximally entangled state is a bipartite state with:

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center)]
  \node (a) at (0,0) {$\eta$};
  \node (b) at (1,0) {$\eta$};
  \draw (a) -- (b);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center)]
  \node (a) at (0,0) {$\eta$};
  \node (b) at (1,0) {$\eta$};
  \draw (a) -- (b);
\end{tikzpicture}
\end{align*}
\]

In a compact dagger category, a state is maximally entangled if and only if it is part of a dagger duality.
Maximally entangled states

In a compact dagger category, a maximally entangled state is a bipartite state with:

\[
\eta \eta = \eta \eta
\]

In a compact dagger category, a state is maximally entangled if and only if it is part of a dagger duality.

Proof:
Dagger duals unique up to unitary

Given dagger duals \((L \vdash R, \eta, \varepsilon)\) and \((L \vdash R', \eta', \varepsilon')\), construct an isomorphism \(R \simeq R'\) as before:

Then:

\[
\eta' \varepsilon \varepsilon' \eta = \eta \varepsilon \varepsilon' \eta' \eta = \eta \eta = \frac{25}{37}
\]
Maximally entangled states unique up to unitary

In a compact dagger category, any two maximally entangled states $I \xrightarrow{\eta, \eta'} A \otimes B$ have a unique unitary $A \xrightarrow{f} A$ with:

\[
\begin{array}{c}
\begin{array}{c}
\eta \\
\end{array}
\end{array}
\xrightarrow{f}
\begin{array}{c}
\begin{array}{c}
\eta'
\end{array}
\end{array}
\]

So maximally entangled states are unique up to a unique unitary.
In a compact dagger category, every morphism satisfies \((f^*)^\dagger = (f^\dagger)^*\).

Proof:

\[
\begin{align*}
(f^*)^\dagger &= \left(\begin{array}{c}
\vdots \\
\end{array}\right)^\dagger \\
&= f(f^\dagger)^* \\
&= f
\end{align*}
\]
Conjugation

On a compact dagger category, conjugation \((-)_*\) is defined as the composite of the dagger and the right-duals functor:

\[
(-)_* := (-)^\dagger = (-)^\dagger^*
\]

Since taking daggers is the identity on objects we have \(A_* := A^*\).
Conjugation

On a compact dagger category, conjugation \((-)\) is defined as the composite of the dagger and the right-duals functor:

\[(\neg)^* := (\neg)^* \dagger = (\neg) \dagger^*\]

Since taking daggers is the identity on objects we have \(A^* := A^*\).

Draw conjugation by flipping the morphism about a vertical axis:

\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\end{array} \\
:= \\
\begin{array}{c}
\text{f}^* \\
\downarrow \\
\end{array}
\]

Since \((-)^*\) and \(\dagger\) are contravariant, \((-)\) is covariant.
Conjugation: examples

Our examples $\text{FHilb}$, $\text{Mat}_C$ and $\text{Rel}$ are all compact dagger categories

- In $\text{FHilb}$: conjugation functor gives conjugate of linear map

- In $\text{Mat}_C$: conjugation functor gives the conjugate of a matrix, each matrix entry replaced by its conjugate as a complex number

- In $\text{Rel}$: conjugation is identity
Trace and dimension

In a compact dagger category, the trace of a morphism $A \xrightarrow{f} A$ is the following scalar $\text{Tr}_A(f)$:

\[ f \]

The dimension of an object $A$ is the scalar $\text{dim}(A) := \text{Tr}_A(\text{id}_A)$.

The trace in $\mathcal{FHilb}$ is the ordinary trace.
Trace and dimension

In a compact dagger category, the trace of a morphism $A \xrightarrow{f} A$ is the following scalar $\text{Tr}_A(f)$:

The dimension of an object $A$ is the scalar $\text{dim}(A) := \text{Tr}_A(\text{id}_A)$. 
Trace and dimension

In a compact dagger category, the trace of a morphism $A \xrightarrow{f} A$ is the following scalar $\operatorname{Tr}_A(f)$:

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (f) at (0,0) {$f$};
    \draw [->] (f) -- (f)弧; \\
    \end{tikzpicture}
\end{array}
\]

The dimension of an object $A$ is the scalar $\dim(A) := \operatorname{Tr}_A(\operatorname{id}_A)$.

The trace in $\mathbf{FHilb}$ is the ordinary trace.
Trace is cyclic

In any compact dagger category, $\text{Tr}_A(g \circ f) = \text{Tr}_B(f \circ g)$.

Proof:

The $g$ slides around the circle, and ends up underneath the $f$. 

31 / 37
Trace and dimension properties

In a compact dagger category:

- $\text{Tr}_I(s) = s$
- $\text{Tr}_{A \otimes B}(f \otimes g) = \text{Tr}_A(f) \circ \text{Tr}_B(g)$
- $\left( \text{Tr}_A(f) \right)^\dagger = \text{Tr}_A(f^\dagger)$

Hence:

- $\text{dim}(I) = \text{id}_I$
- $\text{dim}(A \otimes B) = \text{dim}(A) \circ \text{dim}(B)$
- $A \cong B \Rightarrow \text{dim}(A) = \text{dim}(B)$
Trace and dimension properties

In a compact dagger category:

- $\text{Tr}_I(s) = s$
- $\text{Tr}_{A \otimes B}(f \otimes g) = \text{Tr}_A(f) \circ \text{Tr}_B(g)$
- $(\text{Tr}_A(f))^\dagger = \text{Tr}_A(f^\dagger)$

Hence:

- $\dim(I) = \id_I$
- $\dim(A \otimes B) = \dim(A) \circ \dim(B)$
- $A \simeq B \Rightarrow \dim(A) = \dim(B)$
Dual objects are finite-dimensional

Infinite-dimensional Hilbert spaces do not have duals.

Proof: Similarly we could prove $\dim(A \oplus B) = \dim(A) + \dim(B)$. Suppose $H$ is an infinite-dimensional Hilbert space. Then there is an isomorphism $H \oplus \mathbb{C} \simeq H$. If $H$ had a dual, then $\dim(H) + 1 = \dim(H)$. But this is a contradiction, since there is no complex number with that property.
Dual objects are finite-dimensional

Infinite-dimensional Hilbert spaces do not have duals.

Proof: Similarly we could prove \( \dim(A \oplus B) = \dim(A) + \dim(B) \).
Suppose \( H \) is an infinite-dimensional Hilbert space. Then there is an isomorphism \( H \oplus \mathbb{C} \simeq H \). If \( H \) had a dual, then \( \dim(H) + 1 = \dim(H) \). But this is a contradiction, since there is no complex number with that property.

This argument would not apply in \( \text{Rel} \), since there \( \text{id}_1 + \text{id}_1 = \text{id}_1 \). Indeed, any set has a dual in \( \text{Rel} \), even infinite ones.
Information flow

In well-pointed monoidal dagger category $f = g : A \to B$ if and only if

\[
\begin{array}{c}
\triangleleft b \\
\downarrow \\
\square f \\
\downarrow \\
\triangleleft a
\end{array}
= 
\begin{array}{c}
\triangleleft b \\
\downarrow \\
\square g \\
\downarrow \\
\triangleleft a
\end{array}
\]

for all $a, b : I \to B$: can compare ‘matrix entries’
Information flow

In well-pointed monoidal dagger category \( f = g : A \to B \) if and only if

\[
\begin{array}{c}
\begin{array}{c}
\triangle \quad b \\
\downarrow \quad f \\
\downarrow \quad a
\end{array}

= \quad
\begin{array}{c}
\begin{array}{c}
\triangle \quad b \\
\downarrow \quad g \\
\downarrow \quad a
\end{array}
\end{array}
\end{array}
\]

for all \( a, b : I \to B \): can compare ‘matrix entries’

In \textbf{Rel} can conveniently decorate wires with elements: scalar

\[
\begin{array}{c}
\begin{array}{c}
R \quad z
\end{array}

\begin{array}{c}
\downarrow \\
x
\end{array}

\begin{array}{c}
S \\
\downarrow \\
y
\end{array}
\end{array}
\]

is 1 if and only if there is \( y \) such that following scalars both 1:

\[
\begin{array}{c}
\begin{array}{c}
R \\
\downarrow \\
x
\end{array}

\begin{array}{c}
\downarrow \\
y
\end{array}

\begin{array}{c}
S \\
\downarrow \\
z
\end{array}
\end{array}
\]

and
Information flow

In well-pointed monoidal dagger category $f = g : A \to B$ if and only if

\[
\begin{align*}
\begin{tikzpicture}[scale=0.8]
  \node (a) at (0,0) {$a$};
  \node (b) at (1.5,1.5) {$b$};
  \node (f) at (0.75,0.75) {$f$};
  \draw (a) -- (f) -- (b);
\end{tikzpicture}
\quad =
\quad \begin{tikzpicture}[scale=0.8]
  \node (a) at (0,0) {$a$};
  \node (b) at (1.5,1.5) {$b$};
  \node (g) at (0.75,0.75) {$g$};
  \draw (a) -- (g) -- (b);
\end{tikzpicture}
\end{align*}
\]

for all $a, b : I \to B$: can compare ‘matrix entries’

So can decorate

\[
\begin{tikzpicture}[scale=0.8]
  \node (R) at (0,0) {$R$};
  \node (S) at (0,-2) {$S$};
  \node (x) at (-1,-2) {$x$};
  \node (y) at (-1,-1) {$y$};
  \node (z) at (1,-1) {$z$};
  \draw (R) -- (S);
  \draw (S) -- (x);
  \draw (S) -- (y);
  \draw (R) -- (z);
\end{tikzpicture}
\]

to signify that if $x$ is connected to $z$, then must ‘flow’ through some $y$
In \textbf{FHilb}, have (destructive) interference:
if $g = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, $f = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $x = z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then

\[
\begin{align*}
z & \quad g \quad (1 \ 0) \quad z \\
g & \quad f \quad (1 \ 0) \\
x & \quad f \quad (0 \ 1) \\
\end{align*}
\]

\[
\begin{align*}
&= g + f \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -4 + 4 = 0
\end{align*}
\]

but both histories in the sum are possible
Cups are entangled

If $L \rightarrow R$, and $I \xrightarrow{\eta} R \otimes L$ is a product state, then $\text{id}_L$ and $\text{id}_R$ disconnect (factor through $I$)
Cups are entangled

If $L \rightarrow R$, and $I \xrightarrow{\eta} R \otimes L$ is a product state, then $\text{id}_L$ and $\text{id}_R$ disconnect (factor through $I$)

Proof: Suppose $\eta$ is $I \xrightarrow{\lambda_i^{-1}} I \otimes I \xrightarrow{r \otimes l} R \otimes L$. Then:

\[
\begin{align*}
L & \xrightarrow{\lambda_i^{-1}} I \otimes I \xrightarrow{r \otimes l} R \otimes L.
\end{align*}
\]
Cups are entangled

If $L \rightarrow R$, and $I \xrightarrow{\eta} R \otimes L$ is a product state, then $\text{id}_L$ and $\text{id}_R$ disconnect (factor through $I$)

Proof: Suppose $\eta$ is $I \xrightarrow{\lambda_i^{-1}} I \otimes I \xrightarrow{r \otimes l} R \otimes L$. Then:

Interpreting diagram as history of events, disconnect means output independent of input: $L$ degenerate
Summary

- Dual objects: bend wires, maximally entangled states
- Names and conames: encode morphisms as states
- Dual morphisms: sliding, functorial
- Teleportation: quantum, one-time pad
- Graphical calculus for compact dagger categories: orientation
- Conjugation: combine duals with dagger
- Trace and dimension: turn morphisms into scalars
- Information flow: entanglement vs disconnect