Categories and Quantum Informatics: Scalars

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Many aspects of linear algebra can be described using categorical structures. This chapter examines abstractions of the base field, and inner products.

2.1 Scalars

If we begin with the monoidal category **Hilb**, we can extract from it much of the structure of the complex numbers. The monoidal unit object I is given by the complex numbers $\mathbb{C}$, and so morphisms $I \longrightarrow I$ are linear maps $\mathbb{C} \xrightarrow{f} \mathbb{C}$. Such a map is determined by $f(1)$, since by linearity we have $f(a) = a \cdot f(1)$. So, we have a correspondence between morphisms of type $I \longrightarrow I$ and the complex numbers. Also, it's easy to check that multiplication of complex numbers corresponds to composition of their corresponding linear maps.

In general, it is often useful to think of the the morphisms of type $I \longrightarrow I$ in a monoidal category as behaving like a field in linear algebra. For this reason, we give them a special name.

Definition 2.1. In a monoidal category, the *scalars* are the morphisms $I \longrightarrow I$.

A *monoid* is a set A equipped with a multiplication operation, which we write as juxtaposition of elements of A, and a chosen unit element $1 \in A$, satisfying for all $u, v, w \in A$ an associativity law $u(vw) = (uv)w$ and a unit law $1v = v = v1$. We will study monoids closely from a categorical perspective later, but for now we note that it is easy to show from the axioms of a category that the scalars form a monoid under composition.

Example 2.2. The monoid of scalars is very different in each of our running example categories.

- In **Hilb**, scalars $\mathbb{C} \xrightarrow{f} \mathbb{C}$ correspond to complex numbers $f(1) \in \mathbb{C}$ as discussed above. Composition of scalars $\mathbb{C} \xrightarrow{f,g} \mathbb{C}$ corresponds to multiplication of complex numbers, as $(g \circ f)(1) = g(f(1)) = f(1) \cdot g(1)$. Hence the scalars in **Hilb** are the complex numbers under multiplication.
- In **Set**, scalars are functions $\{\bullet\} \xrightarrow{f} \{\bullet\}$. There is only one unique such function, namely $\mathrm{id}_{\{\bullet\}} \colon \bullet \mapsto \bullet$, which we will also simply write as 1. Hence the scalars in **Set** form the trivial one-element monoid.
- In **Rel**, scalars are relations $\{\bullet\} \xrightarrow{R} \{\bullet\}$. There are two such relations: $F = \emptyset$ and $T = \{(\bullet, \bullet)\}$. Working out the composition in **Rel** gives the following multiplication table:

	F	T
F	F	F
T	F	T

Hence we can recognize the scalars in **Rel** as the Boolean truth values {true, false} under conjunction.

Commutativity

Multiplication of complex numbers is commutative: $ab = ba$. It turns out that this holds for scalars in any monoidal category.

Lemma 2.3. *In a monoidal category, the scalars are commutative.*

Proof. Consider the following diagram, for any two scalars $I \xrightarrow{a,b} I$:

$$\begin{tikzcd}
I \arrow[rrrr, "a"] \arrow[drr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & & & & I \arrow[drr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & & \\
& & I \arrow[rrrr, "a"] & & & & I \\
I \otimes I \arrow[rrrr, "a \otimes \mathrm{id}_I"'] \arrow[drr, "\mathrm{id}_I \otimes b"'] & & & & I \otimes I \arrow[drr, "\mathrm{id}_I \otimes b"'] & & \\
& & I \otimes I \arrow[rrrr, "a \otimes \mathrm{id}_I"'] \arrow[uu, "\lambda_I"', "\rho_I"] & & & & I \otimes I \arrow[uu, "\lambda_I"', "\rho_I"]
\end{tikzcd} \tag{2.1}$$

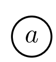
The four side cells of the cube commute by naturality of λ_I and ρ_I, and the bottom cell commutes by an application of the interchange law. Hence we have $ab = ba$. Note the importance of coherence here, as we rely on the fact that $\rho_I = \lambda_I$. $\square$

Example 2.4. The scalars in our example categories are indeed commutative.

- In **Hilb**: multiplication of complex numbers is commutative.
- In **Set**: $1 \circ 1 = 1 \circ 1$ is trivially commutative.
- In **Rel**: let a, b be Boolean values; then (a and b) is true precisely when (b and a) is true.

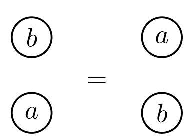
Graphical calculus

We draw scalars as circles:



(2.2)

Commutativity of scalars then has the following graphical representation:



(2.3)

The diagrams are isotopic, so it follows from correctness of the graphical calculus that scalars are commutative. Once again, a nontrivial property of monoidal categories follows straightforwardly from the graphical calculus.

Scalar multiplication

Objects in an arbitrary monoidal category do not have to be anything particularly like vector spaces, at least at first glance. Nevertheless, many of the features of the mathematics of vector spaces can be mimicked. For example, if $a \in \mathbb{C}$ is a scalar and f a linear map, then af is again a linear map, and we can mimic this in general monoidal categories as follows.

Definition 2.5 (Left scalar multiplication)**.** In a monoidal category, for a scalar $I \xrightarrow{a} I$ and a morphism $A \xrightarrow{f} B$, the *left scalar multiplication* $A \xrightarrow{a \bullet f} B$ is the following composite:

$$\begin{tikzcd} A \arrow[r, "a \bullet f"] \arrow[d, "\lambda_A^{-1}"'] & B \\ I \otimes A \arrow[r, "a \otimes f"] & I \otimes B \arrow[u, "\lambda_B"'] \end{tikzcd} \tag{2.4}$$

This abstract scalar multiplication satisfies many properties we are familiar with from scalar multiplication of vector spaces, as the following lemma explores.

Lemma 2.6 (Scalar multiplication)**.** *In a monoidal category, let $I \xrightarrow{a,b} I$ be scalars, and $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be arbitrary morphisms. Then the following properties hold:*

(a) $\mathrm{id}_I \bullet f = f$*;*

(b) $a \bullet b = a \circ b$*;*

(c) $a \bullet (b \bullet f) = (a \bullet b) \bullet f$*;*

(d) $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$*.*

Proof. These statements all follow straightforwardly from the graphical calculus, thanks to the correctness theorem. We also give the direct algebraic proofs. Part (a) follows directly from naturality of λ. For part (b), diagram (2.1) shows that $a \circ b = \lambda_I \circ (a \otimes b) \circ \lambda_I^{-1} = a \bullet b$. Part (c) follows from the following diagram that commutes by coherence:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\lambda_A^{-1}"] & A \arrow[rr, "a \bullet (b \bullet f)"] \arrow[d, "\lambda_A^{-1}"] & & B \arrow[r, "\mathrm{id}_B"] & B \\ I \otimes A \arrow[r, "\mathrm{id}_{I \otimes A}"] \arrow[ddr, "\lambda_I^{-1} \otimes \mathrm{id}_A"'] & I \otimes A \arrow[rr, "a \otimes (b \bullet f)"] \arrow[d, "\mathrm{id}_I \otimes \lambda_A^{-1}"] & & I \otimes B \arrow[r, "\mathrm{id}_{I \otimes B}"] \arrow[u, "\lambda_B"] & I \otimes B \arrow[u, "\lambda_B"'] \\ & I \otimes (I \otimes A) \arrow[rr, "a \otimes (b \otimes f)"] \arrow[d, "\alpha_{I,I,A}^{-1}"] & & I \otimes (I \otimes B) \arrow[u, "\mathrm{id}_I \otimes \lambda_B"] & \\ & (I \otimes I) \otimes A \arrow[rr, "(a \otimes b) \otimes f"] & & (I \otimes I) \otimes B \arrow[u, "\alpha_{I,I,B}"] \arrow[uur, "\lambda_I \otimes \mathrm{id}_B"'] & \end{tikzcd}$$

Part (d) follows from the interchange law. □

Example 2.7. Scalar multiplication looks as follows in our example categories.

- In **Hilb**: if $a \in \mathbb{C}$ is a scalar and $H \xrightarrow{f} K$ a morphism, then $H \xrightarrow{a \bullet f} K$ is the morphism $v \mapsto af(v)$.
- In **Set**, scalar multiplication is trivial: if $A \xrightarrow{f} B$ is a function, and 1 is the unique scalar, then $\mathrm{id}_1 \bullet f = f$ is again the same function.
- In **Rel**: for any relation $A \xrightarrow{R} B$, we find that $\texttt{true} \bullet R = R$, and $\texttt{false} \bullet R = \emptyset$.

2.2 Daggers

In our definition of the category of Hilbert spaces, one aspect seemed strange: inner products are not used in a central way. This leaves a gap in our categorical model, since inner products play a central role in quantum theory. In this section we will see how inner products can be described abstractly using a *dagger functor*, a contravariant involutive endofunctor on the category that is compatible with the monoidal structure. The motivation is the construction of the adjoint of a linear map between Hilbert spaces, which as we will see encodes all the information about the inner products.

Dagger categories

To describe inner products abstractly, begin by thinking about *adjoints*. Any bounded linear map $H \xrightarrow{f} K$ between Hilbert spaces has a unique adjoint, which is another bounded linear map $K \xrightarrow{f^\dagger} H$. We can encode this action as a functor.

Definition 2.8. On **Hilb**, the functor *taking adjoints* $\dagger\colon \mathbf{Hilb} \to \mathbf{Hilb}$ is the contravariant functor that takes objects to themselves, and morphisms to their adjoints as bounded linear maps.

For $\dagger$ to be a contravariant functor it must satisfy the equation $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$and send identities to identities, which is indeed the case for this operation. Furthermore it is the identity on objects, meaning that ${\mathrm{id}_H}^\dagger = \mathrm{id}_H$ for all objects H, and it is involutive, meaning that $(f^\dagger)^\dagger = f$ for all morphisms f.

Knowing all adjoints suffices to reconstruct the inner products on Hilbert spaces. To see how this works, let $\mathbb{C} \xrightarrow{v,w} H$ be states of some Hilbert space H. The following calculation shows that the scalar $\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}$ is equal to the inner product $\langle v|w\rangle$:

$$
\begin{aligned}
(\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}) &\equiv v^\dagger(w(1)) \\
&= \langle 1|v^\dagger(w(1))\rangle \\
&= \langle v|w\rangle
\end{aligned}
\tag{2.5}
$$

So the functor taking adjoints contains all the information required to reconstruct the inner products on our Hilbert spaces. Since we used the inner products to define this functor in the first place, we see that knowing the functor taking adjoints is *equivalent* to knowing the inner products.

This suggests a way to generalize the idea of 'inner products' to arbitrary categories, using the following structure.

Definition 2.9 (Dagger functor, dagger category)**.** A *dagger functor* on a category **C** is an involutive contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ that is the identity on objects. A *dagger category* is a category equipped with a dagger functor.

A contravariant functor is therefore a dagger functor exactly when it has the following properties:

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \tag{2.6}$$

$${\mathrm{id}_H}^\dagger = \mathrm{id}_H \tag{2.7}$$

$$(f^\dagger)^\dagger = f \tag{2.8}$$

The identity-on-objects and contravariant properties mean that if $A \xrightarrow{f} B$, we must have $B \xrightarrow{f^\dagger} A$. The involutive property says that $(f^\dagger)^\dagger = f$.

The canonical dagger functor on **Hilb** is the functor taking adjoints. **Rel** also has a canonical dagger functor.

Definition 2.10. The dagger structure on **Rel** is given by relational converse: for $S \xrightarrow{R} T$, define $T \xrightarrow{R^\dagger} S$ by setting $t\,R^\dagger\,s$ if and only if $s\,R\,t$.

The category **Set** cannot be made into a dagger category: writing $|A|$ for the cardinality of a set A, the set of functions $\mathbf{Set}(A,B)$ contains $|B|^{|A|}$ elements, whereas $\mathbf{Set}(B,A)$ contains $|A|^{|B|}$ elements. A dagger functor would give an bijection between these sets for all A and B, which is not possible.

Another important non-example is **Vect**, the category of complex vector spaces and linear maps. For an infinite-dimensional complex vector space V, the set $\mathbf{Vect}(\mathbb{C}, V)$ has a strictly smaller cardinality than the set $\mathbf{Vect}(V, \mathbb{C})$, so no dagger functor is possible. The category **FVect** containing only finite-dimensional objects *can* be equipped with a dagger functor: one way to do this is by assigning an inner product to every object, and then constructing the associated adjoints. However, it does not come with a *canonical* dagger functor.

A one-object dagger category is also called an *involutive monoid*. It consists of a set M together with an element $1 \in M$ and functions $M \times M \xrightarrow{\cdot} M$ and $M \xrightarrow{\dagger} M$ satisfying $1m = m = m1$, $m(no) = (mn)o$, and $(m^\dagger)^\dagger = m$ for all $m, n, o \in M$.

In a dagger category we give special names to some basic properties of morphisms. These generalize the terms usually reserved for bounded linear maps between Hilbert spaces.

Definition 2.11. A morphism $A \xrightarrow{f} B$ in a dagger category is:

- the *adjoint* of $B \xrightarrow{g} A$ when $g = f^\dagger$;
- *self-adjoint* when $f = f^\dagger$ (and $A = B$);
- *idempotent* when $f = f \circ f$ (and $A = B$);
- a *projection* when it is idempotent and self-adjoint;
- *unitary* when both $f^\dagger \circ f = \mathrm{id}_A$ and $f \circ f^\dagger = \mathrm{id}_B$;
- an *isometry* when $f^\dagger \circ f = \mathrm{id}_A$;
- a *partial isometry* when $f^\dagger \circ f$ is a projection;
- *positive* when $f = g^\dagger \circ g$ for some morphism $A \xrightarrow{g} C$ (and $A = B$).

If a category carries an important structure, it is often fruitful to require that the constructions one makes are compatible with that structure. The dagger functor is an important structure for us, and for most of this book we will require compatibility with it. In the search for good definitions, it is useful to see this as a sort of guiding principle, which we summarize as the *way of the dagger*.

Monoidal dagger categories

We start by looking at cooperation between dagger structure and monoidal structure. For matrices $H_1 \xrightarrow{f_1} K_1$ and $H_2 \xrightarrow{f_2} K_2$, their tensor product $f_1 \otimes f_2$ is given by the Kronecker product, and their adjoints $f_1^\dagger, f_2^\dagger$ are given by conjugate transpose. The order of these two operations is irrelevant: $(f_1 \otimes f_2)^\dagger = f_1^\dagger \otimes f_2^\dagger$. We abstract this behaviour of linear maps to arbitrary monoidal categories.

Definition 2.12 (Monoidal dagger category, braided, symmetric)**.** A *monoidal dagger category* is a dagger category that is also monoidal, such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f and g, and such that all components of the natural isomorphisms α, λ and ρ are unitary. A *braided monoidal dagger category* is a monoidal dagger category equipped with a unitary braiding. A *symmetric monoidal dagger category* is a braided monoidal dagger category for which the braiding is a symmetry.

Example 2.13. Both **Hilb** and **Rel** are symmetric monoidal dagger categories.

- In **Hilb**, we have $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ since the former is the unique map satisfying

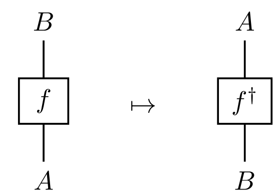
$$\begin{aligned}
&\langle (f \otimes g)^\dagger (v_1 \otimes w_1) | v_2 \otimes w_2 \rangle \\
&\quad = \langle v_1 \otimes w_1 | (f \otimes g)(v_2 \otimes w_2) \rangle \\
&\quad = \langle v_1 \otimes w_1 | f(v_2) \otimes g(w_2) \rangle \\
&\quad = \langle v_1 | f(v_2) \rangle \langle w_1 | g(w_2) \rangle \\
&\quad = \langle f^\dagger(v_1) | v_2 \rangle \langle g^\dagger(w_1) | w_2 \rangle \\
&\quad = \langle (f^\dagger \otimes g^\dagger)(v_1 \otimes w_1) | v_2 \otimes w_2 \rangle .
\end{aligned}$$

- In **Rel**, a simple calculation for $A \xrightarrow{R} B$ and $C \xrightarrow{S} D$ shows that

$$\begin{aligned}
(R \times S)^\dagger &= \{ \big((b,d), (a,c) \big) \mid aRb,\ cSd \} \\
&= R^\dagger \times S^\dagger .
\end{aligned}$$

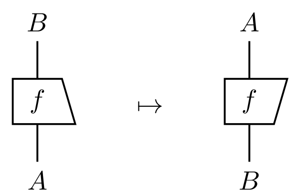
In each case the coherence isomorphisms $\lambda, \rho, \alpha, \sigma$ are also clearly unitary.

We depict taking daggers in the graphical calculus by flipping the graphical representation about a horizontal axis as follows.



(2.9)

To help differentiate between these morphisms, we will draw morphisms in a way that breaks their symmetry. Taking daggers then has the following representation.



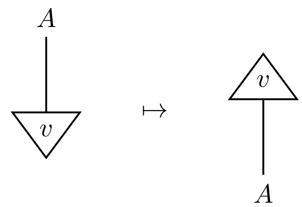
(2.10)

We no longer write the † symbol within the label, as this is now indicated by the orientation of the wedge.

For example, the graphical representation unitarity (see Definition 2.11) is:

(2.11)

In particular, in a monoidal dagger category, we can use this notation for morphisms $I \xrightarrow{v} A$ representing a state. This gives a representation of the adjoint morphism $A \xrightarrow{v^\dagger} I$ as follows.

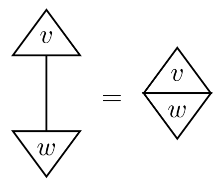


(2.12)

We have described how a state of an object $I \xrightarrow{a} A$ can be thought of as a *preparation* of A by the process a. Dually, a costate $A \xrightarrow{a^\dagger} I$ models the *effect* of eliminating A by the process $a^\dagger$. A dagger functor gives a correspondence between states and effects.

Equation (2.5) demonstrated how to recover inner products from the ability to take daggers of states. Applying this argument graphically yields the following expression for the inner product $\langle v|w\rangle$ of two states $I \xrightarrow{v,w} H$.

$\langle v|w\rangle$ =



(2.13)

The right-hand side picture is defined by this equation. Notice that it is a rotated form of Dirac's bra-ket notation given on the left-hand side. For this reason, we can think of the graphical calculus for monoidal dagger categories as a generalized Dirac notation.

Probabilities

If $I \xrightarrow{v} A$ is a state and $A \xrightarrow{x} I$ an effect, recall that we interpret the scalar $I \xrightarrow{v} A \xrightarrow{x} I$ as the *amplitude* of measuring outcome $x^\dagger$ immediately after preparing state v; in bra-ket notation this would be $\langle x|v\rangle$. The *probability* that this history occurred is the square of its absolute value, which would be $|\langle x^\dagger|v\rangle|^2 = \langle v|x^\dagger\rangle \cdot \langle x^\dagger|v\rangle = \langle v|x^\dagger \circ x(v)\rangle$ in bra-ket notation. This makes sense for abstract scalars, as follows.

Definition 2.14 (Probability)**.** If $I \xrightarrow{v} A$ is a state, and $A \xrightarrow{x} I$ an effect, in a monoidal dagger category, set

$$\mathrm{Prob}(x, v) = v^\dagger \circ x^\dagger \circ x \circ v \colon I \longrightarrow I. \tag{2.14}$$

Example 2.15. In our example categories, probabilities match with our interpretation.

- In **Hilb**, probabilities are non-negative real numbers $|\langle x|v\rangle|^2$.
- In **Rel**, the probability of observing an effect $X \subseteq A$ after preparing the state $V \subseteq A$ is the scalar true when $X \cap V \neq \emptyset$, and the scalar false when X and V are disjoint. This matches with our interpretation that the state V consists of all those elements of A that the initial state $\bullet$ before preparation can possibly evolve into.