Exercise 2.1. Show that the following defines a dagger category:

- **objects** $(A, p)$ are finite sets $A$ equipped with prior probability distributions, functions $p: A \to \mathbb{R}^+$ such that $\sum_{a \in A} p(a) = 1$;
- **morphisms** $(A, p) \xrightarrow{f} (B, q)$ are conditional probability distributions, functions $f: A \times B \to \mathbb{R}^\geq$ such that for all $a \in A$ we have $\sum_{b \in B} f(a, b) = 1$, and for all $b \in B$ we have $q(b) = \sum_{a \in A} p(a) f(a, b)$;
- **composition** is composition of probability distributions as matrices of real numbers;
- **the dagger** is the Bayesian converse, acting on $f: A \times B \to \mathbb{R}^\geq$ to give $f^\dagger: B \times A \to \mathbb{R}^\geq$, defined as $f^\dagger(b, a) = f(a, b)q(b)/p(a)$.

Note that the Bayesian converse is always well-defined since we require our prior probability distributions to be nonzero at every point.

Exercise 2.2. Show that all joint states are product states when $A \otimes B$ is a product of $A$ and $B$. Conclude that monoidal categories modeling nonlocal correlation such as entanglement must have a tensor product that is not a (categorical) product.

Exercise 2.3. Let $A \xrightarrow{R} B$ be a morphism in the dagger category $\text{Rel}$.

- (a) Show that $R$ is unitary if and only if it is (the graph of) a bijection;
- (b) Show that $R$ is self-adjoint if and only if it is symmetric;
- (c) Show that $R$ is positive if and only if $R$ is symmetric and $a R b \Rightarrow a R a$.
- (d) Is every isometry $A \to A$ in $\text{Rel}$ unitary?

Exercise 2.4. Is $\text{Mat}_C$ a monoidal dagger category?

Exercise 2.5. Fuglede’s theorem is the following statement for morphisms $f, g: A \to A$ in $\text{Hilb}$: if $f \circ f^\dagger = f^\dagger \circ f$ and $f \circ g = g \circ f$, then also $f^\dagger \circ g = g \circ f^\dagger$. Show that this does not hold in $\text{Rel}$. 

Categories and Quantum Informatics exercise sheet 3:

 Scalars