Categories and Quantum Informatics exercise sheet 3: Scalars

Exercise 2.1. Show that the following defines a dagger category:

- *objects* (A, p) are finite sets A equipped with *prior probability distributions*, functions $p : A \longrightarrow \mathbb{R}^{+}$ such that $\sum_{a\in A} p(a) = 1$;
- *morphisms* $(A, p) \xrightarrow{f} (B, q)$ are *conditional probability distributions*, functions $f : A \times B \longrightarrow \mathbb{R}^{\geq 0}$ such that for all $a \in A$ we have $\sum_{b\in B} f(a, b) = 1$, and for all $b \in B$ we have $q(b) = \sum_{a\in A} p(a)f(a, b)$;
- *composition* is composition of probability distributions as matrices of real numbers;
- the *dagger* is the *Bayesian converse*, acting on $f : A \times B \longrightarrow \mathbb{R}^{\geq 0}$ to give $f^{\dagger} : B \times A \longrightarrow \mathbb{R}^{\geq 0}$, defined as $f^{\dagger}(b, a) = f(a, b)p(a)/q(b)$.

Note that the Bayesian converse is always well-defined since we require our prior probability distributions to be nonzero at every point.

Exercise 2.2. Show that all joint states are product states when $A \otimes B$ is a product of A and B and I is a terminal object. Conclude that monoidal categories modeling nonlocal correlation such as entanglement must have a tensor product that is not a (categorical) product.

Exercise 2.3. Let $A \xrightarrow{R} B$ be a morphism in the dagger category **Rel**.

(a) Show that R is unitary if and only if it is (the graph of) a bijection;

(b) Show that R is self-adjoint if and only if it is symmetric;

(c) Show that R is positive if and only if R is symmetric and $a\,R\,b \Rightarrow a\,R\,a$.

(d) Is every isometry $A \longrightarrow A$ in **Rel** unitary?

Exercise 2.4. Is $\mathbf{Mat}_{\mathbb{C}}$ a monoidal dagger category?

Exercise 2.5. Fuglede's theorem is the following statement for morphisms $f, g \colon A \longrightarrow A$ in **Hilb**: if $f \circ f^{\dagger} = f^{\dagger} \circ f$ and $f \circ g = g \circ f$, then also $f^{\dagger} \circ g = g \circ f^{\dagger}$. Show that this does not hold in **Rel**.

Exercise 2.6. Recall the notion of local equivalence from Exercise Sheet 2. In **Hilb**, we can write a state $\mathbb{C} \xrightarrow{\phi} \mathbb{C}^2 \otimes \mathbb{C}^2$ as a column vector

$$\phi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

or as a matrix

$$M_{\phi} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) Show that ϕ is an entangled state if and only if M_{ϕ} is invertible. (Hint: a matrix is invertible if and only if it has nonzero determinant.)

(b) Show that $M_{(\mathrm{id}_{\mathbb{C}^2} \otimes f)\circ\phi} = M_{\phi} \circ f^{\mathrm{T}}$, where $\mathbb{C}^2 \xrightarrow{f} \mathbb{C}^2$ is any linear map and f^{T} is the transpose of f in the canonical basis of $\mathbb{C}^2$.

(c) Use this to show that there are three families of locally equivalent joint states of $\mathbb{C}^2 \otimes \mathbb{C}^2$.