# Categories and Quantum Informatics exercise sheet 3: Scalars

**Exercise 2.1.** The composition of two morphisms is a well-defined morphisms. The dagger is well-defined and involutive, and respects composition.

**Exercise 2.2.** We will show that we can write any state $\psi : I \to A \otimes B$ as the product state $\psi = (p_1 \circ \psi \otimes p_1 \circ \psi) \circ \lambda_I^{-1}$. Since the tensor product is a categorical product, $\psi : I \to A \times B$ makes the diagram below comute. The map $< p_1 \circ \psi, p_2 \circ \psi > \circ \lambda^{-1}$ makes the diagram commute as well for the following reason: Since $I \cong I \times I$ and $I$ is the terminal object, $I \times I$ is the terminal object; hence, there is one unique arrow $I \to I \times I$, so $\lambda^{-1}$ makes the lower triangle commute. By definition of the product, $< p_1 \circ \psi, p_2 \circ \psi >$ makes the upper square commute. It follows from the universal property of products that $\psi =< p_1 \circ \psi, p_2 \circ \psi > \circ \lambda^{-1}$.

$$\begin{tikzcd}
A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
I \arrow[u, "p_1 \circ \psi"] & I \times I \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[u, dashed] & I \arrow[u, "p_2 \circ \psi"'] \\
& I \arrow[ul, equal] \arrow[ur, equal] \arrow[u, dashed] &
\end{tikzcd}$$

**Exercise 2.3.** (a) First, $R^\dagger \circ R = \mathrm{id}_A$ implies that $R$ relates every element $a$ of $A$ to some element of $B$. If it was related to two elements of $B$, that would violate $R \circ R^\dagger = \mathrm{id}_B$. Finally, $R \circ R^\dagger = \mathrm{id}_B$ means that every element of $B$ is related to some element of $A$. So all in all, $R$ relates each element of $A$ to precisely one element of $B$, and vice versa.

(b) By definition, $R$ being self-adjoint means that $aRb$ if and only if $aR^\dagger b$, which in turns holds if and only if $bRa$.

(c) If $R$ is symmetric and satisfies $a\,R\,b \Rightarrow a\,R\,a$, setting

$$S = \{(a, (x, y)) \mid a \in A, (x, y) \in R, a = x \text{ or } a = y\}$$

gives $R = S^\dagger \circ S$.

(d) No; $R = \{(\bullet, 0), (\bullet, 1)\} \colon \{\bullet\} \longrightarrow \{0, 1\}$ satisfies $R^\dagger \circ R = \mathrm{id}_{\{\bullet\}}$, but is not (the graph of) a subset inclusion.

**Exercise 2.4.** Transposition gives a dagger, and the Kronecker product of matrices respects transposition.

**Exercise 2.5.** Take $A = \{0, 1\}$, $R = \{(0, 0), (1, 0), (1, 1)\}$, and $S = \{(1, 0)\}$. Then $R^\dagger \circ R = R \circ R^\dagger$ and $R \circ S = S \circ R$, but not $R^\dagger \circ S = S \circ R^\dagger$.

**Exercise 2.6.** (a) Notice that $\phi$ is the product state of $\mathbb{C} \xrightarrow{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)} \mathbb{C}^2$ and $\mathbb{C} \xrightarrow{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} \mathbb{C}^2$ precisely when

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ux \\ uy \\ vx \\ vy \end{pmatrix}.$$

In this case, $\det(M_\phi) = ad - bc = uxvy - uyvx = 0$, so $\phi$ is invertible.
Conversely, suppose $ad - bc = 0$. If $a \neq 0$, then we may take $u = 1$, $v = ca^{-1}$, $x = a$, and $y = b$ to show that $\phi$ is a product state. Similar choices work when one of $b$,$c$ or $d$ is nonzero. Finally, if $a = b = c = d = 0$, we may take $u = v = x = y = 0$.

(b) Compute

$$M_\phi \circ f^{\mathrm{T}} = \begin{pmatrix} au + bv & ax + by \\ cu + dv & cx + dy \end{pmatrix},$$

and

$$(\mathrm{id}_{\mathbb{C}^2} \otimes f) \circ \phi = \begin{pmatrix} u & v & 0 & 0 \\ x & y & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & x & y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} au + bv \\ ax + by \\ cu + dv \\ cx + dy \end{pmatrix}.$$

(c) First, we show that all entangled states $\phi$ are locally equivalent to $\psi = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right)$. Indeed, if $M_\phi$ is invertible, then $M_\psi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = M_\phi \circ (((M_\phi)^{-1})^{\mathrm{T}})^{\mathrm{T}} = M_{(\mathrm{id}_{\mathbb{C}^2} \otimes (M_\phi^{-1})^{\mathrm{T}}) \circ \phi}$, so $\psi = (\mathrm{id}_{\mathbb{C}^2} \otimes (M_\phi^{-1})^{\mathrm{T}} \circ \phi$.
Also, product states can never be locally equivalent to entangled states, so all entangled states form one equivalence class.
Second, the zero state $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ is an equivalence class of its own: if any state is locally equivalent to the zero state, then it must have been the zero state to begin with.
Third, we show that all nonzero product states are locally equivalent. Indeed, if states $\left(\begin{smallmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{smallmatrix}\right)$ are nonzero, there exist invertible maps taking $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ to $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ to $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$.