Categories and Quantum Informatics

Week 3: Scalars

Chris Heunen
Monoidal structure of \texttt{Hilb} encodes structure of complex numbers.

- As a set: \texttt{Hilb}(\mathbb{C}, \mathbb{C}), endomorphisms of tensor unit.
- Multiplication: of complex numbers is given by composition.
- Commutativity: \( ab = ba \) for all elements of \texttt{Hilb}(\mathbb{C}, \mathbb{C}).

A \texttt{scalar} in a monoidal category is a morphism \( I \rightarrow I \).

Can replicate a lot of linear algebra in any monoidal category.
**Lemma**: In a monoidal category, scalars commute.

**Proof.** Consider the following diagram, for any two scalars $I \xrightarrow{a,b} I$:

Side cells: naturality of $\lambda_I$ and $\rho_I$. Bottom cell: interchange law. Vertical arrows: coherence.
Graphical calculus

We draw a scalar $I \xrightarrow{a} I$ as a circle:

$\bigcirc$
Graphical calculus

We draw a scalar \( I \xrightarrow{a} I \) as a circle:

\[
\begin{array}{c}
\text{a} \\
\end{array}
\]

Commutativity of scalars becomes:

\[
\begin{array}{c}
\text{b} & \text{a} \\
\text{a} & \text{b} \\
\end{array}
\]

Diagrams are isotopic, so it follows from correctness of the graphical calculus that scalars are commutative.
Scalar multiplication

Can multiply linear map $H \xrightarrow{f} J$ with number $c \in \mathbb{C}$, to get $H \xrightarrow{c \cdot f} J$. Works in any monoidal category.

The left scalar multiplication of morphism $A \xrightarrow{f} B$ with scalar $I \xrightarrow{a} I$ is

Graphically:
Scalar multiplication

Many familiar properties. For $I \xrightarrow{a,b} I$ and $A \xrightarrow{f} B, B \xrightarrow{g} C$:

- $\text{id}_I \cdot f = f$
- $a \cdot b = a \circ b$
- $a \cdot (b \cdot f) = (a \cdot b) \cdot f$
- $(b \cdot g) \circ (a \cdot f) = (b \circ a) \cdot (g \circ f)$

**Proof.** Use graphical calculus.
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**Proof.** Use graphical calculus. 

- In **Hilb**: if $a \in \mathbb{C}$ is a scalar and $H \xrightarrow{f} K$ a morphism, then $H \xrightarrow{af} K$ is the morphism $\nu \mapsto af(\nu)$.
- In **Set**, scalar multiplication is trivial: if $A \xrightarrow{f} B$ is a function, then $\text{id}_1 \bullet f = f$ is again the same function.
- In **Rel**: for any relation $A \xrightarrow{R} B$, $\text{true} \bullet R = R$, and $\text{false} \bullet R = \emptyset$. 
Daggers

In the definition of \( \textbf{FHilb} \), something was a bit strange: we didn’t use the inner products at all.

Inner products give adjoint linear maps:

\[
(g \circ f)^\dagger = f^\dagger \circ g^\dagger \quad \text{id}_H^\dagger = \text{id}_H \quad (f^\dagger)^\dagger = f
\]

Taking adjoints: contravariant involutive functor, identity on objects.
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Conversely, can recover inner products from this functor:

\[(\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}) \equiv v^\dagger(w(1)) = \langle 1|v^\dagger(w(1))\rangle = \langle v|w\rangle\]

So $\dagger$ and $\langle -| - \rangle$ encode equivalent information.
Dagger categories

A **dagger** on a category $\mathbf{C}$ is an involutive contravariant functor $\dagger : \mathbf{C} \to \mathbf{C}$ that is the identity on objects. A **dagger category** is a category equipped with a dagger.

Examples:

- **Hilb** is a dagger category using adjoint linear maps.
- **Mat$_\mathbb{C}$** is a dagger category using the conjugate transpose.
- **Rel** can be given a dagger functor by relational converse: for $S \xrightarrow{R} T$, define $T \xrightarrow{R^\dagger} S$ by setting $t R^\dagger s$ if and only if $s R t$.
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- **Set** cannot be made into a dagger category: $\text{Set}(A, B)$ has size $|B|^{|A|}$, while $\text{Set}(B, A)$ has size $|A|^{|B|}$.
- **Vect** cannot be given a dagger functor: $\text{Vect}(\mathbb{C}, V)$ has a smaller cardinality than $\text{Vect}(V, \mathbb{C})$ when $V$ is infinite-dimensional.
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- **FVect** can be given dagger (e.g. by assigning an inner product to objects and constructing adjoints.) But not canonically so.
Terminology

A morphism $A \xrightarrow{f} B$ in a dagger category is:

- the adjoint of $B \xrightarrow{g} A$ when $g = f^\dagger$
- self-adjoint when $f = f^\dagger$
- a projection when $f = f^\dagger$ and $f \circ f = f$
- unitary when both $f^\dagger \circ f = \text{id}_A$ and $f \circ f^\dagger = \text{id}_B$
- an isometry when $f^\dagger \circ f = \text{id}_A$
- a partial isometry when $f^\dagger \circ f$ is a projection
- positive when $f = g^\dagger \circ g$ for some morphism $H \xrightarrow{g} K$
Graphical calculus

Depict taking daggers by reflection in horizontal axis.

\[
\begin{array}{c}
B \\
| \\
| \\
\downarrow \\
A \\
\end{array}
\quad \xrightarrow{\dagger} \\
\begin{array}{c}
A \\
| \\
| \\
\downarrow \\
B \\
\end{array}
\]
Graphical calculus

Depict taking daggers by reflection in horizontal axis.

To differentiate, draw morphisms in a way that breaks symmetry. We also drop the label $\dagger$ from the morphism box.
States, effects, scalars

Dagger gives a correspondence between states and effects:

\[ A \downarrow v \quad \mapsto \quad \uparrow v \quad A \downarrow \]
States, effects, scalars

Dagger gives a correspondence between states and effects:

\[ A \xrightarrow{\dagger} A \]

Inner product between two states:

\[ \langle v | w \rangle = \begin{array}{c} v \\ \downarrow \\ w \end{array} = \begin{array}{c} v \\ \downarrow \\ w \end{array} \]

Generalised form of Dirac’s bra-ket notation.
Way of the dagger

A **monoidal dagger category** is a dagger category that is also monoidal, such that:

- \((f \otimes g)^\dagger = f^\dagger \otimes g^\dagger\) for all morphisms \(f\) and \(g\);
- the natural isomorphisms \(\alpha\), \(\lambda\) and \(\rho\) are unitary at every stage.

A **braided monoidal dagger category** is a monoidal dagger category equipped with a unitary braiding.

A **symmetric monoidal dagger category** is a braided monoidal dagger category for which the braiding is a symmetry.
Summary

- Scalars: morphisms \(I \rightarrow I\)
- Scalars commute
- Scalar multiplication
- Daggers: generalise inner product
- Way of the dagger: monoidal dagger categories