Categories and Quantum Informatics

Week 3: Scalars

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Scalars

Monoidal structure of **Hilb** encodes structure of complex numbers.

- As a set: $\mathbf{Hilb}(\mathbb{C}, \mathbb{C})$, endomorphisms of tensor unit.
- Multiplication: of complex numbers is given by composition.
- Commutativity: $ab = ba$ for all elements of $\mathbf{Hilb}(\mathbb{C}, \mathbb{C})$.

A scalar in a monoidal category is a morphism $I \to I$.

Can replicate a lot of linear algebra in any monoidal category.

Scalars commute

Lemma: In a monoidal category, scalars commute.
Proof. Consider the following diagram, for any two scalars $I \xrightarrow{a,b} I$:

$$
\begin{tikzcd}
I \arrow[rr, "a"] \arrow[dr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & & I \arrow[dr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & \\
& I \arrow[rr, "a"] & & I \\
I \otimes I \arrow[rr, "a \otimes \mathrm{id}_I"'] \arrow[dr, "\mathrm{id}_I \otimes b"'] & & I \otimes I \arrow[dr, "\mathrm{id}_I \otimes b"'] & \\
& I \otimes I \arrow[uu, "\lambda_I", "\rho_I"'] \arrow[rr, "a \otimes \mathrm{id}_I"'] & & I \otimes I \arrow[uu, "\lambda_I", "\rho_I"']
\end{tikzcd}
$$

Side cells: naturality of λ_I and ρ_I. Bottom cell: interchange law.
Vertical arrows: coherence. $\square$

Graphical calculus

We draw a scalar $I \xrightarrow{a} I$ as a circle:

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We draw a scalar $I \xrightarrow{a} I$ as a circle:

$$a$$

Commutativity of scalars becomes:

$$\begin{matrix} b \\ \\ a \end{matrix} \quad = \quad \begin{matrix} a \\ \\ b \end{matrix}$$

Diagrams are isotopic, so it follows from correctness of the graphical calculus that scalars are commutative.

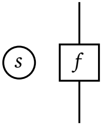
Scalar multiplication

Can multiply linear map $H \xrightarrow{f} J$ with number $c \in \mathbb{C}$, to get $H \xrightarrow{c \cdot f} J$.
Works in any monoidal category.

The left scalar multiplication of morphism $A \xrightarrow{f} B$ with scalar $I \xrightarrow{a} I$ is

$$\begin{tikzcd} A \arrow[r, "a \bullet f"] \arrow[d, "\lambda_A^{-1}"'] & B \\ I \otimes A \arrow[r, "a \otimes f"] & I \otimes B \arrow[u, "\lambda_B"'] \end{tikzcd}$$

Graphically:



Scalar multiplication

Many familiar properties. For $I \xrightarrow{a,b} I$ and $A \xrightarrow{f} B$, $B \xrightarrow{g} C$:

- $\mathrm{id}_I \bullet f = f$
- $a \bullet b = a \circ b$
- $a \bullet (b \bullet f) = (a \bullet b) \bullet f$
- $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$

Proof. Use graphical calculus. □

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- In **Hilb**: if $a \in \mathbb{C}$ is a scalar and $H \xrightarrow{f} K$ a morphism, then $H \xrightarrow{a \bullet f} K$ is the morphism $v \mapsto af(v)$.
- In **Set**, scalar multiplication is trivial: if $A \xrightarrow{f} B$ is a function, then $\mathrm{id}_1 \bullet f = f$ is again the same function.
- In **Rel**: for any relation $A \xrightarrow{R} B$, $\mathtt{true} \bullet R = R$, and $\mathtt{false} \bullet R = \emptyset$.

Daggers

In the definition of **FHilb**, something was a bit strange:
we didn't use the inner products at all.

Inner products give adjoint linear maps:

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \qquad \mathrm{id}_H{}^\dagger = \mathrm{id}_H \qquad (f^\dagger)^\dagger = f$$

Taking adjoints: contravariant involutive functor, identity on objects.

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Conversely, can *recover* inner products from this functor:

$$(\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}) \equiv v^\dagger(w(1)) = \langle 1 | v^\dagger(w(1)) \rangle = \langle v | w \rangle$$

So $\dagger$ and $\langle - | - \rangle$ encode *equivalent* information.

Dagger categories

A dagger on a category $\mathbf{C}$ is an involutive contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ that is the identity on objects. A dagger category is a category equipped with a dagger.

Examples:

- **Hilb** is a dagger category using adjoint linear maps.
- $\mathbf{Mat}_{\mathbb{C}}$ is a dagger category using the conjugate transpose.
- **Rel** can be given a dagger functor by relational converse: for $S \xrightarrow{R} T$, define $T \xrightarrow{R^\dagger} S$ by setting $t\, R^\dagger\, s$ if and only if $s\, R\, t$.

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- **Set** cannot be made into a dagger category: $\mathbf{Set}(A,B)$ has size $|B|^{|A|}$, while $\mathbf{Set}(B,A)$ has size $|A|^{|B|}$.
- **Vect** cannot be given a dagger functor: $\mathbf{Vect}(\mathbb{C},V)$ has a smaller cardinality than $\mathbf{Vect}(V,\mathbb{C})$ when V is infinite-dimensional.

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- **FVect** *can* be given dagger (e.g. by assigning an inner product to objects and constructing adjoints.) But not *canonically* so.

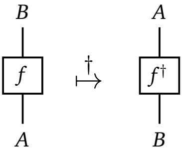
Terminology

A morphism $A \xrightarrow{f} B$ in a dagger category is:

- the adjoint of $B \xrightarrow{g} A$ when $g = f^\dagger$
- self-adjoint when $f = f^\dagger$
- a projection when $f = f^\dagger$ and $f \circ f = f$
- unitary when both $f^\dagger \circ f = \mathrm{id}_A$ and $f \circ f^\dagger = \mathrm{id}_B$
- an isometry when $f^\dagger \circ f = \mathrm{id}_A$
- a partial isometry when $f^\dagger \circ f$ is a projection
- positive when $f = g^\dagger \circ g$ for some morphism $H \xrightarrow{g} K$

Graphical calculus

Depict taking daggers by reflection in horizontal axis.



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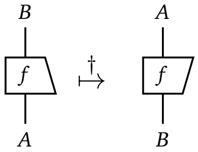
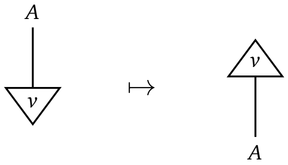


To differentiate, draw morphisms in a way that breaks symmetry.
We also drop the label † from the morphism box.

States, effects, scalars

Dagger gives a correspondence between states and effects:



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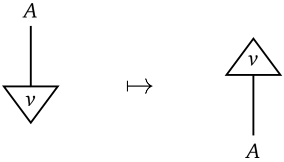
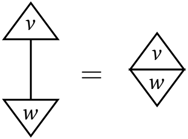


Inner product between two states:

$$\langle v|w\rangle =$$



Generalised form of Dirac's bra-ket notation.

Way of the dagger

A monoidal dagger category is a dagger category that is also monoidal, such that:

- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f and g;
- the natural isomorphisms α, λ and ρ are unitary at every stage.

A braided monoidal dagger category is a monoidal dagger category equipped with a unitary braiding.

A symmetric monoidal dagger category is a braided monoidal dagger category for which the braiding is a symmetry.

Summary

- Scalars: morphisms $I \to I$
- Scalars commute
- Scalar multiplication
- Daggers: generalise inner product
- Way of the dagger: monoidal dagger categories