Categories and Quantum Informatics exercise sheet 2: Hilbert spaces, monoidal categories

Exercise 1.1. Show that in **FHilb**, the isomorphisms are precisely the bijective morphisms.

Exercise 1.2. Prove that direct sums form products and coproducts in **FHilb**.

Exercise 1.3. Show that the Kronecker product of matrices f, g, and h, satisfies $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

Exercise 1.4. Let A, B, C, D be objects in a monoidal category. Construct a morphism

$$(((A \otimes I) \otimes B) \otimes C) \otimes D \longrightarrow A \otimes (B \otimes (C \otimes (I \otimes D))).$$

Can you find another?

Exercise 1.5. Convert the following algebraic equations into graphical language. Which would you expect to be true in any symmetric monoidal category?

(a) $(g \otimes \mathrm{id}) \circ \sigma \circ (f \otimes \mathrm{id}) = (f \otimes \mathrm{id}) \circ \sigma \circ (g \otimes \mathrm{id})$ for $A \xrightarrow{f,g} A$.

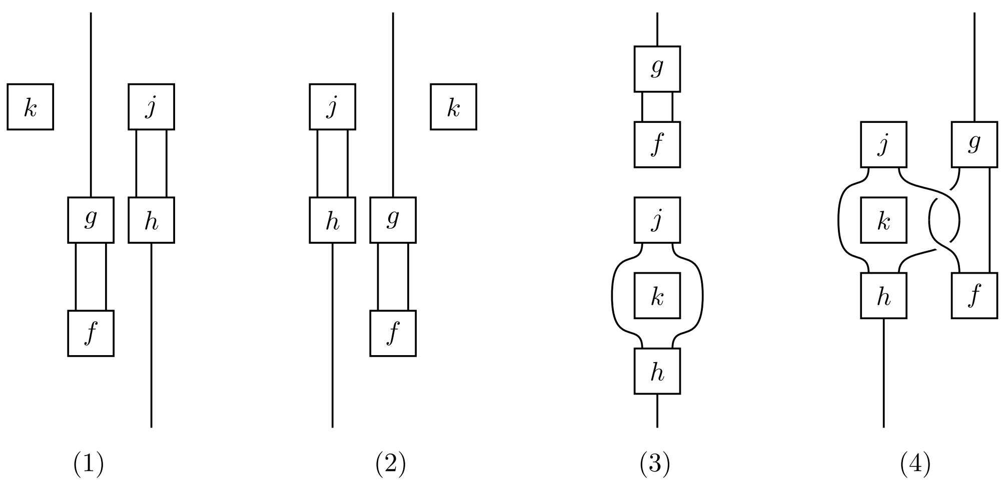
(b) $(f \otimes (g \circ h)) \circ k = (\mathrm{id} \otimes f) \circ ((g \otimes h) \circ k)$, for $A \xrightarrow{k} B \otimes C$, $C \xrightarrow{h} B$ and $B \xrightarrow{f,g} B$.

(c) $(\mathrm{id} \otimes h) \circ g \circ (f \otimes \mathrm{id}) = (\mathrm{id} \otimes f) \circ g \circ (h \otimes \mathrm{id})$, for $A \xrightarrow{f,h} A$ and $A \otimes A \xrightarrow{g} A \otimes A$.

(d) $h \circ (\mathrm{id} \otimes \lambda) \circ (\mathrm{id} \otimes (f \otimes \mathrm{id})) \circ (\mathrm{id} \otimes \lambda^{-1}) \circ g = h \circ g \circ \lambda \circ (f \otimes \mathrm{id}) \circ \lambda^{-1}$, for $A \xrightarrow{g} B \otimes C$, $I \xrightarrow{f} I$ and $B \otimes C \xrightarrow{h} D$.

(e) $\rho_C \circ (\mathrm{id} \otimes f) \circ \alpha_{C,A,B} \circ (\sigma_{A,C} \otimes \mathrm{id}_B) = \lambda_C \circ (f \otimes \mathrm{id}) \circ \alpha^{-1}_{A,B,C} \circ (\mathrm{id} \otimes \sigma_{C,B}) \circ \alpha_{A,C,B}$ for $A \otimes B \xrightarrow{f} I$.

Exercise 1.6. Consider the following diagrams in the graphical calculus:

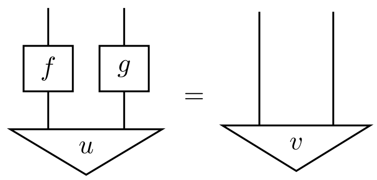


(a) Which of the diagrams (1), (2) and (3) are equal as morphisms in a monoidal category?

(b) Which of the diagrams (1), (2), (3) and (4) are equal as morphisms in a braided monoidal category?

(c) Which of the diagrams (1), (2), (3) and (4) are equal as morphisms in a symmetric monoidal category?

Exercise 1.7. We say that two joint states $I \xrightarrow{u,v} A \otimes B$ are *locally equivalent*, written $u \sim v$, if there exist invertible maps $A \xrightarrow{f} A$, $B \xrightarrow{g} B$ such that



(a) Show that $\sim$ is an equivalence relation.

(b) Find all isomorphisms $\{0,1\} \longrightarrow \{0,1\}$ in **Rel**.

(c) Write out all 16 states of the object $\{0,1\} \times \{0,1\}$ in **Rel**.

(d) Use your answer to (b) to group the states of (c) into locally equivalent families. How many families are there? Which of these are entangled?

Exercise 1.8. Complete the following proof that $\rho_I = \lambda_I$ in a monoidal category, by labelling every arrow, and indicating for each region whether it follows from the triangle equation, the pentagon equation, naturality, or invertibility. Head-to-tail arrows are always inverse pairs.