Categories and Quantum Informatics exercise sheet 2 answers:
Hilbert spaces, monoidal categories

**Exercise 1.1.** Clearly isomorphisms are bijective morphisms. Any bijective morphism \( f: H \to K \) has a set-theoretical inverse \( f^{-1}: K \to H \); we have to prove that it is a morphism. It is additive: \( f^{-1}(x + y) = f^{-1}(f(f^{-1}(x)) + f(f^{-1}(y))) = f^{-1}(f(f^{-1}(x) + f^{-1}(y))) = f^{-1}(x) + f^{-1}(y) \). It respects scalar multiplication: \( f^{-1}(sx) = f^{-1}(sf(f^{-1}(x))) = f^{-1}(f(sf^{-1}(x))) = sf^{-1}(x) \). (And it is automatically bounded as \( H \) and \( K \) are finite-dimensional.)

**Exercise 1.2.** The projections and injections are given by

\[
\begin{align*}
 p_H &: H \oplus K \to H & p_H(x, y) &= x \\
 p_K &: H \oplus K \to H & p_K(x, y) &= y \\
 i_H &: H \to H \oplus K & i_H(x) &= (x, 0) \\
 i_K &: H \to H \oplus K & i_K(y) &= (0, y)
\end{align*}
\]

and the universal properties

\[
\begin{align*}
 (f, g) &: A \to B & (f, g)(a) &= (f(a), g(a)) \\
 h &: H \to K & h(x) &= (x, 0)
\end{align*}
\]

are given by

\[
\begin{align*}
 (f, g) &: A \to B & (f, g)(a) &= (f(a), g(a)) \\
 h &: H \to K & h(x) &= (x, 0)
\end{align*}
\]

**Exercise 1.3.** Both \((f \otimes g) \otimes h\) and \(f \otimes (g \otimes h)\) expand to

\[
\begin{align*}
 (f_{11} g_{11} h_{11}, f_{11} g_{11} h_{12}, f_{11} g_{12} h_{11}, f_{11} g_{12} h_{12}, f_{12} g_{11} h_{11}, f_{12} g_{11} h_{12}, f_{12} g_{12} h_{11}, f_{12} g_{12} h_{12}) \\
 (f_{11} g_{21} h_{21}, f_{11} g_{21} h_{22}, f_{11} g_{22} h_{21}, f_{11} g_{22} h_{22}, f_{12} g_{21} h_{21}, f_{12} g_{21} h_{22}, f_{12} g_{22} h_{21}, f_{12} g_{22} h_{22}) \\
 (f_{11} g_{31} h_{31}, f_{11} g_{31} h_{32}, f_{11} g_{32} h_{31}, f_{11} g_{32} h_{32}, f_{12} g_{31} h_{31}, f_{12} g_{31} h_{32}, f_{12} g_{32} h_{31}, f_{12} g_{32} h_{32}) \\
 (f_{11} g_{41} h_{41}, f_{11} g_{41} h_{42}, f_{11} g_{42} h_{41}, f_{11} g_{42} h_{42}, f_{12} g_{41} h_{41}, f_{12} g_{41} h_{42}, f_{12} g_{42} h_{41}, f_{12} g_{42} h_{42}) \\
 (f_{11} g_{51} h_{51}, f_{11} g_{51} h_{52}, f_{11} g_{52} h_{51}, f_{11} g_{52} h_{52}, f_{12} g_{51} h_{51}, f_{12} g_{51} h_{52}, f_{12} g_{52} h_{51}, f_{12} g_{52} h_{52})
\end{align*}
\]
Note: we might try to take cardinal numbers $\mu, \nu, \ldots$ as objects in $\text{Mat}_C$, and $\mu$-by-$\nu$ matrices of complex numbers such that each row and column is square summate as morphisms $\mu \rightarrow \nu$. That is a fine category. But it is not monoidal in the same way as above. Taking $\mu \nu$ as tensor product of objects is fine. The issue is with tensor products of morphisms. To write down the Kronecker product of a $\mu$-by-$\nu$ matrix $f$ and a $\kappa$-by-$\lambda$ matrix $g$, you need functions $\varphi_\mu: \mu \times \kappa \rightarrow \mu$ and $\psi_\kappa: \mu \times \kappa \rightarrow \kappa$ to say $(f \otimes g)_{ij} = f_{\varphi_\mu(i), \varphi_\lambda(j)}g_{\varphi_\kappa(i), \psi_\lambda(j)}$. In the finite case, we took $\varphi_m(i) = \lfloor i/m \rfloor$ and $\psi_m(i) = i \mod m$ (so that $i = m \cdot \varphi_m(i) + \psi_m(i)$). But it seems unlikely that functions $\varphi_\mu$ and $\psi_\mu$ exist for all cardinals $\mu$ that satisfy this associativity.

**Exercise 1.4.** For example:

\[
\begin{array}{c}
\frac{((A \otimes I) \otimes B) \otimes C) \otimes D}{\alpha_{A,B,C} \otimes \text{id}_D} & ((A \otimes B) \otimes C) \otimes D \\
\frac{A \otimes ((B \otimes C) \otimes D)}{\alpha_{A,B,C} \otimes \text{id}_D} & A \otimes ((B \otimes C) \otimes D) \\
\frac{A \otimes (B \otimes (C \otimes D))}{\text{id}_A \otimes \alpha_{B,C,D}} & A \otimes (B \otimes (C \otimes (I \otimes D)))
\end{array}
\]

Because of coherence, this is the only morphism of this type built from the data of a monoidal category. So unless we have more information about the category than just that it’s monoidal, we cannot find another morphism with the same domain and codomain.

**Exercise 1.5.** Recall that the swap map is natural.

(a) Taking $A = \{0, 1\}$, $f = \text{id}_A$ and $g(0) = 1$ and $g(1) = 0$ in $(\text{Set}, \times)$ shows that

(b) Taking $A = C = \{a\}$, $B = \{0, 1\}$, $k(a) = (0, a)$, $f = \text{id}_B$ and $g(0) = 1$, $g(1) = 0$ in $(\text{Set}, \times)$ shows that

(c) Taking $A = \{0, 1\}, g = \text{id}_{A \times A}, f = \text{id}_A$ and $h(0) = 1$, $h(1) = 0$ in $(\text{Set}, \times)$ shows that
The equation presented below:
\[ \begin{array}{c}
\text{f} \\
\text{g}
\end{array} \begin{array}{c}
\text{h} \\
\text{g}
\end{array} = \begin{array}{c}
\text{h} \\
\text{f}
\end{array} \]
is clearly true in the graphical calculus for monoidal categories.

The equation presented below:
\[ \begin{array}{c}
\text{f} \\
\text{g}
\end{array} \begin{array}{c}
\text{h} \\
\text{h}
\end{array} = \begin{array}{c}
\text{f} \\
\text{f}
\end{array} \]
is clearly true in the graphical calculus for symmetric monoidal categories.

Exercise 1.6. (a) In monoidal categories, equalities of the diagrams hold iff they can be continuously deformed into each other using 2-dimensional isotopy. Diagram (1) can be continuously deformed into diagram (2), so they are equal. In diagram (3), the scalar \( k \) is stuck between the wires of \( j \) and \( h \), so it is not equal to (1) and (2).

(b) In braided monoidal categories, equalities of the diagrams hold iff they can be continuously deformed into each other using 3-dimensional isotopy. From (a) we have (1) \( = (2) \). Using 3-dimensional isotopy, we can show (1) \( = (2) = (3) \) by taking the scalar \( k \) out in the third dimension and then moving it over the enclosing wires. However, we can’t show that (4) is equal to the other three diagrams using the axioms of a braided monoidal category. Note, that removing the crossing of the wires in (4) requires that \( \sigma_{B,A} \circ \sigma_{A,B} = \text{id}_{A \otimes B} \) which is always true for symmetric monoidal categories, but not necessarily for braided monoidal categories.

(c) So, in (c) all diagrams are equal, but in (b) (1) \( = (2) = (3) \neq (4) \).

Exercise 1.7. Two joint states are locally equivalent when then can be transformed into one another using only uncorrelated local operations. So if two joint states possess a different ‘amount of correlation’, they will not be locally equivalent.

(a) Taking \( f = g = \text{id} \) shows that \( u \sim u \). If \( u \sim v \) because \( v = (f \otimes g) \circ u \), then also \( (f^{-1} \otimes g^{-1}) \circ v = u \), whence \( v \sim u \). Finally, if \( u \sim v \) and \( v \sim w \) because \( v = (f \otimes g) \circ u \) and \( w = (k \otimes h) \circ v \), then \( w = ((k \circ f) \otimes (h \circ g)) \circ u \) by the interchange law, so that \( u \sim w \).

(b) Isomorphisms in \( \text{Rel} \) are the graphs of bijections: \( \{(0,0),(1,1)\} \) and \( \{(0,1),(1,0)\} \) are the only isomorphisms \( \{0,1\} \rightarrow \{0,1\} \).

(c) States of \( \{0,1\} \times \{0,1\} \) are its subsets.

(d) Simply starting with one state we haven’t classified yet, and generating all possible locally equivalent ones by pre- and/or postcomposing with all bijections, we find the following 7 local equivalence classes.

\[ \emptyset \]
\[ \{\{0,0\}\} \sim \{\{0,1\}\} \sim \{\{1,0\}\} \sim \{\{1,1\}\} \]
\[ \{\{0,0\},\{0,1\}\} \sim \{\{1,0\},\{1,1\}\} \]
\[
{(0, 0), (1, 0)} \sim {(0, 1), (1, 1)} \\
{(0, 0), (1, 1)} \sim {(0, 1), (1, 0)} \\
{(0, 0), (0, 1), (1, 0)} \sim {(0, 0), (0, 1), (1, 1)} \sim {(0, 1), (0, 1), (1, 1)} \\
{(0, 0), (0, 1), (1, 0), (1, 1)} \\
\]

Notice that local equivalence respects cardinality (but states of the same cardinality need not be locally equivalent).

**Exercise 1.8.** The axiom applying to each region can be deduced from its number of non-identity sides: 2 for invertibility, 3 for triangle, 4 for naturality, and 5 for pentagon. To save space below we write \( I \otimes f \) for \( \text{id}_I \otimes f \).