Categories and Quantum Informatics

Week 2: Hilbert spaces, Monoidal categories

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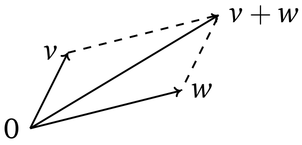
informatics

Vector spaces

Set V with element 0, functions $+\colon V \times V \to V$, and $\cdot\colon \mathbb{C} \times V \to V$

- *additive associativity*: $u + (v + w) = (u + v) + w$;
- *additive commutativity*: $u + v = v + u$;
- *additive unit*: $v + 0 = v$;
- *additive inverses*: there exists a $-v \in V$ such that $v + (-v) = 0$;
- *additive distributivity*: $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$
- *scalar unit*: $1 \cdot v = v$;
- *scalar distributivity*: $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$;
- *scalar compatibility*: $a \cdot (b \cdot v) = (ab) \cdot v$.



Example: $\mathbb{C}^n$

Linear maps

Function $f\colon V \to W$ is linear when

$$f(v+w) = f(v) + f(w)$$
$$f(a \cdot v) = a \cdot f(v)$$

Vector spaces and linear maps form a category **Vect**

Bases and matrices

- Vectors $\{e_i\}$ form basis when any vector v takes the form $v = \sum_i v_i e_i$ for $v_i \in \mathbb{C}$ in precisely one way.
- Any vector space has a basis
any two bases have the same cardinality: dimension
- Finite-dimensional vector spaces and linear maps
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- Finite-dimensional vector spaces and linear maps
  form a category **FVect**
- Given bases $\{d_i\}$ and $\{e_j\}$, linear map $V \xrightarrow{f} W$ gives matrix $f(d_i)_j$,
  and vice versa
- There is a category $\mathbf{Mat}_{\mathbb{C}}$ of natural numbers and matrices
  There is an equivalence $\mathbf{Mat}_{\mathbb{C}} \to \mathbf{FVect}$ given by $n \mapsto \mathbb{C}^n$

Hilbert spaces

Vector space H with inner product $\langle - | - \rangle \colon H \times H \to \mathbb{C}$ such that

- *conjugate-symmetric*: $\langle v|w \rangle = \langle w|v \rangle^*$
- *linear* in second argument:
  $\langle v|a \cdot w \rangle = a \cdot \langle v|w \rangle$ and $\langle u|v + w \rangle = \langle u|v \rangle + \langle u|w \rangle$
- *positive definite*: $\langle v|v \rangle \geq 0$ with equality iff $v = 0$
- *complete* in the norm $\|v\| = \sqrt{\langle v|v \rangle}$
  (if $\sum_{i=1}^{\infty} \|v_i\| < \infty$ then $\lim_n \|v - \sum_{i=1}^{n} v_i\| = 0$ for some v)

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Linear $f \colon H \to K$ is bounded when $\|f(v)\| \leq \|f\| \cdot \|v\|$ for some $\|f\| \in \mathbb{R}$

Hilbert spaces and bounded linear maps form category **Hilb**
Finite-dimensional Hilbert spaces form category **FHilb**

Dual space

- Basis is orthogonal when $\langle e_i | e_j \rangle = 0$ for $i \neq j$; orthonormal if $\langle e_i | e_i \rangle = 1$
- Bounded $H \xrightarrow{f} K$ has adjoint $K \xrightarrow{f^\dagger} H$ with $\langle f(v) | w \rangle = \langle v | f^\dagger(w) \rangle$ (conjugate transpose matrix)
- Given $v \in H$, its ket $\mathbb{C} \xrightarrow{|v\rangle} H$ is $z \mapsto zv$; bra $H \xrightarrow{\langle v|} \mathbb{C}$ is $w \mapsto \langle v | w \rangle$
- Dual Hilbert space H^* is $\mathbf{Hilb}(H, \mathbb{C})$

Tensor products

Function $f\colon U \times V \to W$ is bilinear when it is linear in each variable

Tensor product of vector spaces U and V is a vector space $U \otimes V$ with bilinear $f\colon U \times V \to U \otimes V$ such that for every bilinear $g\colon U \times V \to W$ there exists unique linear $h\colon U \otimes V \to W$ such that $g = h \circ f$

$$\begin{tikzcd} U \times V \arrow[r, "\text{(bilinear) } f"] \arrow[dr, "\text{(bilinear) } g"'] & U \otimes V \arrow[d, dashed, "h \text{ (linear)}"] \\ & W \end{tikzcd}$$

Hilbert space with $\langle u \otimes v | u' \otimes v' \rangle = \langle u | u' \rangle \langle v | v' \rangle$

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Hilbert space with $\langle u \otimes v | u' \otimes v' \rangle = \langle u | u' \rangle \langle v | v' \rangle$

If $H \xrightarrow{f} H'$ and $K \xrightarrow{g} K'$ then $f \otimes g\colon H \otimes K \to H' \otimes K'$

$$(f \otimes g) = \begin{pmatrix} (f_{11}g) & (f_{12}g) & \cdots & (f_{1n}g) \\ (f_{21}g) & (f_{22}g) & \cdots & (f_{2n}g) \\ \vdots & \vdots & \ddots & \vdots \\ (f_{m1}g) & (f_{m2}g) & \cdots & (f_{mn}g) \end{pmatrix}$$

Monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.

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Monoidal category theory adds the idea of parallelism:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of P and Q to construct a proof of the conjunction (P and Q).

Why so serious?

- Let A, B and C be processes, and let $\otimes$ be parallel composition
- What *relationship* should there be between these systems?

$$(A \otimes B) \otimes C \qquad\qquad A \otimes (B \otimes C)$$

- It's not right to say they're *equal*, since even just for sets,

$$(S \times T) \times U \neq S \times (T \times U).$$

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
- What should the relationship be between $A \otimes B$ and $B \otimes A$?

Monoidal category

is a category $\mathbf{C}$ equipped with the following data:

- a tensor product functor

$$\otimes\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C};$$

- a unit object

$$I \in \mathrm{Ob}(\mathbf{C});$$

- an associator natural isomorphism

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C);$$

- a left unitor natural isomorphism

$$I \otimes A \xrightarrow{\lambda_A} A;$$

- and a right unitor natural isomorphism

$$A \otimes I \xrightarrow{\rho_A} A.$$

Monoidal category

must satisfy triangle and pentagon equations:

$$
\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}
$$

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\begin{tikzcd}
\big(A \otimes (B \otimes C)\big) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes \big((B \otimes C) \otimes D\big) \arrow[d, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \\
\big((A \otimes B) \otimes C\big) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[dr, "\alpha_{A\otimes B,C,D}"'] & & A \otimes \big(B \otimes (C \otimes D)\big) \\
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& & (A \otimes B) \otimes (C \otimes D) \arrow[urr, "\alpha_{A,B,C\otimes D}"'] & &
\end{tikzcd}
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Coherence theorem for monoidal categories: If the pentagon and triangle equations hold, so does any well-typed equation built from α, λ, ρ and their inverses. (to appreciate this, try to prove $\lambda_I = \rho_I$!)

Set is monoidal

- tensor product is Cartesian product of sets
- tensor unit is a chosen singleton set $\{\bullet\}$
- associators $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ defined by $\big((a, b), c\big) \mapsto \big(a, (b, c)\big)$
- left unitors $I \times A \xrightarrow{\lambda_A} A$ defined by $(\bullet, a) \mapsto a$
- right unitors $A \times I \xrightarrow{\rho_A} A$ defined by $(a, \bullet) \mapsto a$

Other tensor products exist, this one is canonical for classical theory

Rel is monoidal

- tensor product is Cartesian product of sets
  on morphisms: $(a,c)(R \times S)(b,d)$ if and only if aRb and cSd

- tensor unit is a chosen singleton set $= \{\bullet\}$

- associators $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the relations defined by $\big((a,b),c\big) \sim \big(a,(b,c)\big)$

- left unitors $I \times A \xrightarrow{\lambda_A} A$ are the relations defined by $(\bullet, a) \sim a$

- right unitors $A \times I \xrightarrow{\rho_A} A$ are the relations defined by $(a, \bullet) \sim a$

This is not a categorical product in **Rel**

Hilb is monoidal

- tensor product $\otimes$: $\mathbf{Hilb} \times \mathbf{Hilb} \to \mathbf{Hilb}$ is tensor product
- tensor unit I is the one-dimensional Hilbert space $\mathbb{C}$
- associators $(H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K)$
  defined by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$
- left unitors $\mathbb{C} \otimes H \xrightarrow{\lambda_H} H$ defined by $1 \otimes u \mapsto u$
- right unitors $H \otimes \mathbb{C} \xrightarrow{\rho_H} H$ defined by $u \otimes 1 \mapsto u$

Other tensor products exist, this one is canonical for quantum theory

Interchange

Any morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{h} E$ and $E \xrightarrow{j} F$ in a monoidal category satisfy the interchange law:

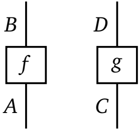
$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$

Proof:

$$\begin{aligned}
(g \circ f) \otimes (j \circ h) &= \otimes(g \circ f, j \circ h) \\
&= \otimes\big((g, j) \circ (f, h)\big) && \text{(composition in } \mathbf{C} \times \mathbf{C}\text{)} \\
&= \big(\otimes(g, j)\big) \circ \big(\otimes(f, h)\big) && \text{(functoriality of } \otimes\text{)} \\
&= (g \otimes j) \circ (f \otimes h)
\end{aligned}$$

Graphical calculus

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, draw $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ as:



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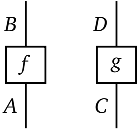

The tensor unit I is drawn as the empty diagram:

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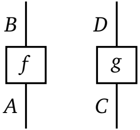


The tensor unit I is drawn as the empty diagram:

Unitors and associators are also not depicted:

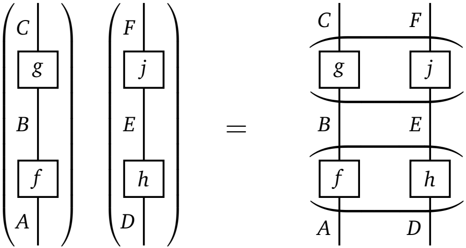
λ_A ρ_A $\alpha_{A,B,C}$

Coherence is essential for the graphical calculus: as there can only be a single morphism built from their components of any given type, it *doesn't matter* that their graphical calculus encodes no information

Graphical calculus

Interchange law trivialises:

$$(g \circ f) \otimes (j \circ h) \quad = \quad (g \otimes j) \circ (f \otimes h)$$

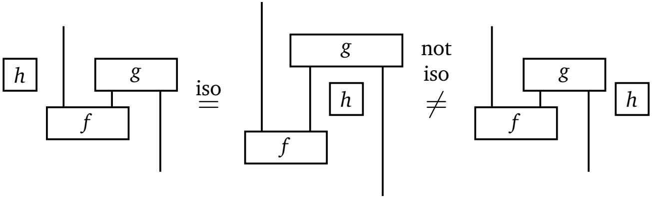


Apparent complexity of monoidal categories just complexity of *geometry of the plane*. In geometrical notation complexity vanishes.

Isotopy

Two diagrams are planar isotopic when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane
- input and output wires terminate at lower and upper boundaries
- components never intersect



(Height of diagrams may change, and input/output wires may slide horizontally along boundary, but may not change order)

Correctness

Theorem: well-formed equation $f = g$ in monoidal category follows from the axioms $\iff$ it holds graphically up to planar isotopy

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Theorem: well-formed equation $f = g$ in monoidal category follows from the axioms $\iff$ it holds graphically up to planar isotopy

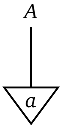
- $P(f, g)$ = 'under the axioms of a monoidal category, $f = g$'
- $Q(f, g)$ = 'graphically, f and g are planar isotopic'

Soundness is the assertion that $P(f, g) \Rightarrow Q(f, g)$ for all such f and g
(easy to prove: just check each axiom)

Completeness is the converse: $Q(f, g) \Rightarrow P(f, g)$ for such f and g
(harder: must show that planar isotopy is generated by finite set of moves, each being implied by the monoidal axioms)

States

Cannot 'look inside' object to see elements, must use morphisms.
A state of an object A is a morphism $I \to A$.



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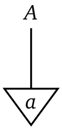
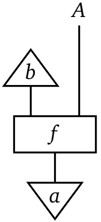


- In **Hilb**: linear functions $\mathbb{C} \to H$, so elements of H
- In **Set**: functions $\{\bullet\} \to A$, so elements of A
- In **Rel**: relations $\{\bullet\} \xrightarrow{R} A$, so subsets of A

Effects

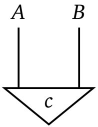
An effect on an object A is a morphism $A \to I$

Interpret effect as *observation* that a system has some property
States, effects, and other morphisms, build up histories:

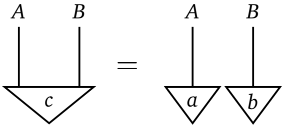


Joint states

A morphism $I \xrightarrow{c} A \otimes B$ is a joint state of A and B.



It is a product state when of the form $I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:



It is entangled when not a product state.

Joint states: examples

- In **Set**:
  - *joint states* of A and B are elements of $A \times B$
  - *product states* are elements $(a, b) \in A \times B$
  - *entangled states* don't exist

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- In **Hilb**:
  - *joint states* of H and K are elements of $H \otimes K$
  - *product states* are factorizable states
  - *entangled states* are entangled states in the quantum sense

Braiding

A braided monoidal category has a natural isomorphism

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satisfying the hexagon equations

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(A \otimes B) \otimes C \arrow[d, "\sigma_{A,B} \otimes \mathrm{id}_C"] & B \otimes (C \otimes A) \arrow[u, "\alpha^{-1}_{B,C,A}"'] \\
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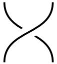
\end{tikzcd}$$

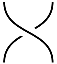
- In **Hilb**: $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ defined by $a \otimes b \mapsto b \otimes a$
- In **Set**: $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ defined by $(a, b) \mapsto (b, a)$
- In **Rel**: $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ defined by $(a, b) \sim (b, a)$

Braiding

We draw the braiding as:



$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$



$$B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B$$

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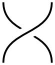

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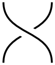
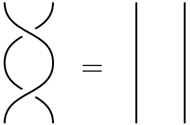
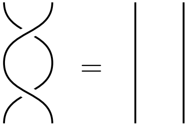


$$B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B$$

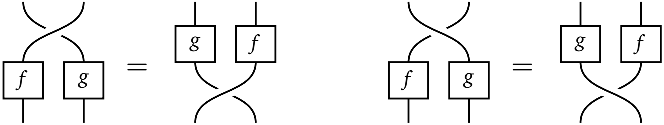
The strands of a braiding cross over each other, so the diagrams are not planar; they are inherently 3-dimensional. Invertibility becomes:





Braiding

Naturality becomes:



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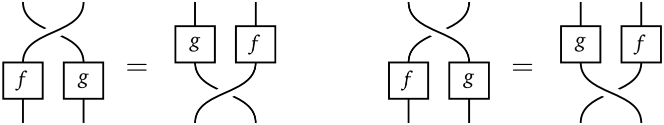
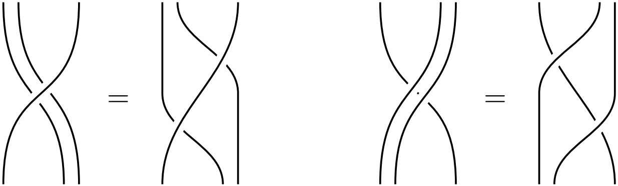
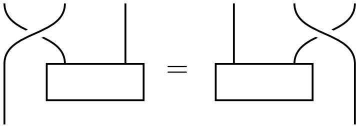


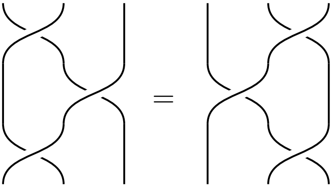
Hexagon equations become:



Graphical calculus

Braided monoidal categories have sound and complete graphical calculus: well-formed equation between morphisms in a braided monoidal category follows from the axioms $\iff$ it holds in the graphical language up to 3-dimensional isotopy.

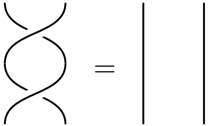




Symmetry

Braided monoidal category is symmetric when

$$\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$$



Strings can pass through each other, no knots: 4d geometry

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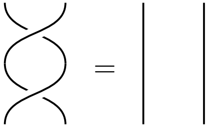
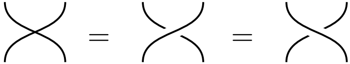


Strings can pass through each other, no knots: 4d geometry

Because $\sigma_{A,B} = \sigma_{B,A}^{-1}$ we may draw



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- But equivalence $\mathbf{FHilb} \simeq \mathbf{Mat}_{\mathbb{C}}$ is monoidal (tensor product $n \otimes m = nm$, tensor unit 1)

Summary

- Category of Hilbert spaces and bounded linear maps
- Monoidal category: coherent tensor products
- Sound and complete graphical calculus
- States and effects: histories
- Braiding and symmetry: correct graphical calculus