Categories and Quantum Informatics

Week 10: ZX-calculus

Chris Heunen

THE UNIVERSITY of EDINBURGH
informatics

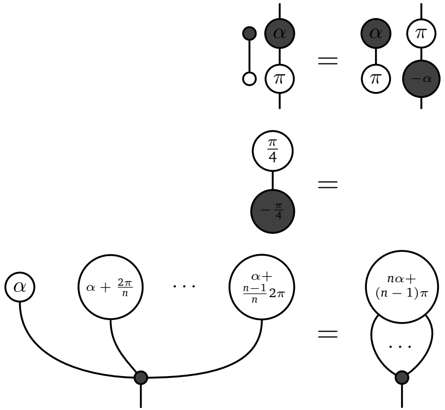
ZX calculus

Can break many-qubit gates into more primitive components.
Adding a couple of rules to complementary observables:

- can describe any possible quantum computation
- manipulating diagrams graphically doesn't change meaning
- any equality of circuits can be proven graphically!

ZX axioms

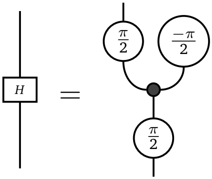
The ZX calculus concerns two strongly complementary classical structures and in a compact dagger category. Phases of integer multiples of $\pi/4$ are allowed, satisfying the following:



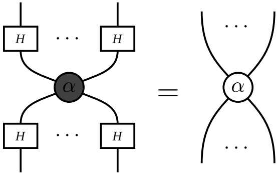
for all $n = 1, 2, 3, \ldots$, as well as their colour-swapped versions.

ZX rules

The Hadamard gate must be definable by:



and satisfy:



as well as the colour-swapped version.

Soundness

Generators and relations give compact dagger subcategory of **FHilb**. The formal symbols have a standard interpretation, written $[\![-]\!]$.

- $[\![$$]\!]: \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ copies the Z basis
- $[\![H]\!] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2} : \mathbb{C}^2 \to \mathbb{C}^2$.

Any graphical manipulations done with ZX diagrams yield valid equalities between matrices under the standard interpretation:

Theorem: Let D_1, D_2 be diagrams in the $\frac{\pi}{4}$-ZX calculus. If D_1 equals D_2 under the axioms of the ZX calculus, then $[\![D_1]\!] = [\![D_2]\!]$.

Approximate universality

Any linear transformation from m qubits to n qubits can be approximated up to arbitrary precision with ZX diagrams:

Theorem: For any morphism $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \xrightarrow{f} \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ in **FHilb**, and any error margin $\varepsilon > 0$, there exists a diagram D in the ZX calculus, that only includes phases that are integer multiples of $\frac{\pi}{4}$, such that $\|[\![D]\!] - f\| < \varepsilon$.

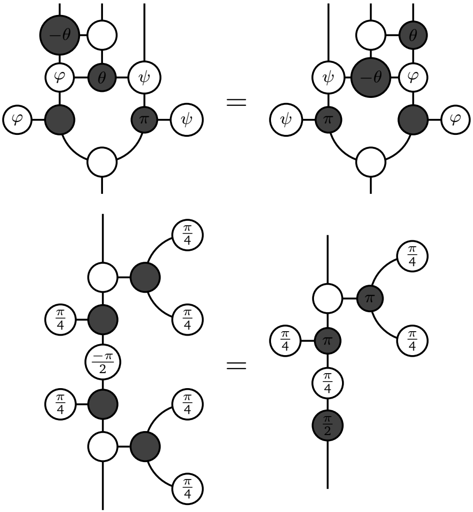
Completeness

Is the ZX calculus *complete*?
If two linear transformations are equal, and are both given by some ZX calculus diagrams, is there always a graphical proof of this using only the axioms of the ZX calculus?

The answer is no when we allow arbitrary phases $\phi \in [0, 2\pi)$.
But if we restrict the angles, the answer is yes!
We can restrict to integer multiples of π, or to integer multiples of $\frac{\pi}{2}$.
However, with these restrictions the ZX calculus is no longer universal, as it might use phases that do not meet the restriction.

$\frac{\pi}{4}$-ZX additional rules

To get completeness for phases that are multiples of $\frac{\pi}{4}$, we need to add the following further two axioms:



for any phases φ, ψ, θ that are multiples of $\frac{\pi}{4}$.

Completeness

Theorem: Let D_1, D_2 be diagrams in the $\frac{\pi}{4}$-ZX calculus. If $[\![D_1]\!] = [\![D_2]\!]$, then $D_1 = D_2$ under the axioms of the $\frac{\pi}{4}$-ZX calculus.