Categories and Quantum Informatics exercise sheet 1: Categorical semantics

Exercise 0.1. Let $(P, \leq)$ be a partially ordered set. Show that the following is a category: the objects are the elements x of P, and there is a unique morphism $x \to y$ if and only if $x \leq y$.

Exercise 0.2. Let M be a *monoid*: a set M together with an associative binary "multiplication" operation $M \times M \to M$ written as $(m, n) \mapsto mn$ and an element $1 \in M$ such that $1m = m = m1$. Show that the following is a category: there is a single object $*$, the morphisms $* \to *$ are elements of M, and composition is multiplication. Conversely, show that any category with a single object comes from a monoid in this way.

Exercise 0.3. Let $G = (V, E)$ be a directed graph. Show that the following is a category: objects are vertices $v \in V$, morphisms $v \to w$ are paths $v \xrightarrow{e_1} \cdots \xrightarrow{e_n} w$ with $e_i \in E$, and composition is concatenation of paths. Choose $n \geq 5$, and draw a graph with n edges whose category has more than n morphisms.

Exercise 0.4.
(a) If P and Q are partially ordered sets regarded as categories, show that functors $P \to Q$ are functions $f \colon P \to Q$ that are monotone: if $x \leq y$ then $f(x) \leq f(y)$.
(b) If M and N are monoids regarded as categories, show that functors $M \to N$ are functions $f \colon P \to Q$ that are homomorphisms: $f(1) = 1$ and $f(mn) = f(m)f(n)$.
(c) If G and H are graphs regarded as categories, what are functors $G \to H$?

Exercise 0.5.
(a) Show that partially ordered sets and monotone functions form a category.
(b) Show that monoids and homomorphisms form a category.

Exercise 0.6.
(a) Show that in **Set**, the isomorphisms are exactly the bijections.
(b) Show that in the category of monoids and homomorphisms, the isomorphisms are exactly the bijective morphisms.
(c) Show that in the category of partially ordered sets and monotone functions, the isomorphisms are not the same as the bijective morphisms.

Exercise 0.7. Consider the following isomorphisms of categories and determine which hold.
(a) $\mathbf{Rel} \simeq \mathbf{Rel}^{\mathrm{op}}$
(b) $\mathbf{Set} \simeq \mathbf{Set}^{\mathrm{op}}$
(c) For a fixed set X with powerset $P(X)$ regarded as a category, $P(X) \simeq P(X)^{\mathrm{op}}$

Exercise 0.8. Let $(P, \leq)$ be a partially ordered set, and regard it as a category.
(a) Show that a product of x and y is a greatest lower bound: an element $x \wedge y$ such that $x \wedge y \leq x$ and $x \wedge y \leq y$, and if any other element satisfies $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.
(b) Show that a coproduct of x and y is a least upper bound.

Exercise 0.9. In any category with binary products, show that $A \times (B \times C) \simeq (A \times B) \times C$.