Categories and Quantum Informatics exercise sheet 1 answers: Categorical semantics

Exercise 0.1. Composition arises from transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$. This is automatically associative. Identities arise from reflexivity: $x \leq x$. (We don't actually need anti-symmetry, pre-orders also induce categories this way.)

Exercise 0.2. Associativity of the composition of the category is precisely associativity of the monoid multiplication.

Note: pre-orders and monoids are two 'extreme' types of categories. Pre-orders have lots of objects and as few morphisms as possible. Monoids have as few objects as possible and lots of morphisms. In a sense any category is a mixture of these two extremes.

Exercise 0.3. Concatenating paths is associative. Identities arise from paths $v \longrightarrow v$ of length 0.

Exercise 0.4.
(a) A functor $P \longrightarrow Q$ by definition consists of a function $f\colon P \longrightarrow Q$ (on objects) that maps morphisms to morphisms. This means precisely that if $x \leq y$ is a morphism in P, then there must be a morphism $f(x) \leq f(y)$ in Q.

(b) A functor $M \longrightarrow N$ by definition consists of a function $\{*\} \longrightarrow \{*\}$ (on objects), and a function $f\colon M \longrightarrow N$ (on morphisms). The latter has to preserve composition ($f(mn) = f(m)f(n)$) and identities ($f(1) = 1$).

(c) Functors $G \longrightarrow H$ by definition consist of a function $f\colon \mathrm{Vertices}(G) \longrightarrow \mathrm{Vertices}(H)$ (on objects), and a function $g\colon \mathrm{Edges}(G) \longrightarrow \mathrm{Paths}(H)$. The latter induces a function $\mathrm{Paths}(G) \longrightarrow \mathrm{Paths}(H)$ that respects associativity of composition and identities by definition of composition and identities in the category G.

Exercise 0.5.
(a) Composition of monotone functions is monotone, and the identity is a monotone function.

(b) Composition of homomorphisms is a homomorphism, and the identity is a homomorphism.

Exercise 0.6.
(a) Isomorphisms are clearly bijections. Conversely, suppose $f\colon A \longrightarrow B$ is a bijection. Then there exists a function $f^{-1}\colon B \longrightarrow A$ with $f(f^{-1}(b)) = b$ and $f(f^{-1}(a)) = a$. So f is an isomorphism.

(b) Isomorphisms are clearly bijective morphisms. Conversely, suppose $f\colon M \longrightarrow N$ is a bijective morphism. Then there exists a function $f^{-1}\colon N \longrightarrow M$ that inverts it. We have to show that f^{-1} is a homomorphism. Clearly $f^{-1}(1) = f^{-1}(f(f^{-1}(1))) = f^{-1}(f(1)) = 1$. Similarly $f^{-1}(xy) = f^{-1}(f(f^{-1}(x))f(f^{-1}(y))) = f^{-1}(f(f^{-1}(x)f^{-1}(y))) = f^{-1}(x)f^{-1}(y)$.

(c) Let P be the partially ordered set $\{0,1\}$ where 0 and 1 are incomparable: $0 \not\leq_P 1$ nor $1 \not\leq_P 0$. Let Q be the set $\{0,1\}$ partially ordered by $0 \leq_Q 1$ (but not $1 \leq_Q 0$). Let $f\colon P \longrightarrow Q$ be the function $f(0) = 0$ and $f(1) = 1$. Then f is bijective and monotone. Its inverse would have to be the set-theoretic function $Q \longrightarrow P$ given by $0 \mapsto 0$ and $1 \mapsto 1$, but that function is not monotone.

Exercise 0.7.
(a) True: the functor that sends a set A to itself, and a relation $R \subseteq A \times B$ to $\{(b,a) \mid (a,b) \in R\} \subseteq B \times A$, is its own inverse.

(b) False: if there were an isomorphism, then $\mathbf{Set}(A,B) \simeq \mathbf{Set}(B,A)$ for any sets A, B. But for e.g.

$A = \{*\}$ and $B = \{0, 1\}$ these two (hom)sets have different cardinality.

(c) True: the assignment on objects that sends $U \in P(X)$ to its complement $X \setminus U \in P(X)$ is functorial, and its own inverse.

Exercise 0.8. (a) A product of x and y is by definition an object $x \wedge y$ such that $x \geq x \wedge y \leq y$. It has to satisfy the universal property: if there is another object z with $x \geq z \leq y$, then there is a (unique) morphism $z \leq x \wedge y$.

(b) Reverse all the inequality signs.

Exercise 0.9. The universal property of $A \times B$ provides a morphism that we'll call $\mathrm{id}_A \times p_B$:

$$\begin{tikzcd} & A \times (B \times C) \arrow[dl, "p_A"'] \arrow[d, dashed, "\mathrm{id}_A \times p_B"] \arrow[dr, "p_B \circ p_{B\times C}"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

The universal property of $(A \times B) \times C$ now provides a morphism $f\colon A \times (B \times C) \longrightarrow (A \times B) \times C$:

$$\begin{tikzcd} & A \times (B \times C) \arrow[dl, "\mathrm{id}_A \times p_B"'] \arrow[d, dashed, "f"] \arrow[dr, "p_C \circ p_{B\times C}"] & \\ A \times B & (A \times B) \times C \arrow[l, "p_{A\times B}"] \arrow[r, "p_C"'] & C \end{tikzcd}$$

Similarly we find a morphism $g\colon (A \times B) \times C \longrightarrow A \times (B \times C)$.
Now $p_A \circ (g \circ f) = p_A \circ \mathrm{id}_{A\times(B\times C)}$ and $p_{B\times C} \circ (g \circ f) = p_{B\times C} \circ \mathrm{id}_{A\times(B\times C)}$. But the universal property of $A \times (B \times C)$ says there is only one morphisms that can satisfy this, so we must have $g \circ f = \mathrm{id}$. Similarly $f \circ g = \mathrm{id}$.