## Categories and Quantum Informatics exercise sheet 1 answers: Categorical semantics

**Exercise 0.1.** Composition arises from transitivity: if  $x \le y$  and  $y \le z$  then  $x \le z$ . This is automatically associative. Identities arise from reflexivity:  $x \le x$ . (We don't actually need anti-symmetry, pre-orders also induce categories this way.)

**Exercise 0.2.** Associativity of the composition of the category is precisely associativity of the monoid multiplication.

Note: pre-orders and monoids are two 'extreme' types of categories. Pre-orders have lots of objects and as few morphisms as possible. Monoids have as few objects as possible and lots of morphisms. In a sense any category is a mixture of these two extremes.

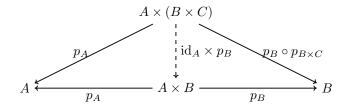
**Exercise 0.3.** Concatenating paths is associative. Identities arise from paths  $v \rightarrow v$  of length 0.

- **Exercise 0.4.** (a) A functor  $P \rightarrow Q$  by definition consists of a function  $f: P \rightarrow Q$  (on objects) that maps morphisms to morphisms. This means precisely that if  $x \leq y$  is a morphism in P, then there must be a morphism  $f(x) \leq f(y)$  in Q.
  - (b) A functor  $M \to N$  by definition consists of a function  $\{*\} \to \{*\}$  (on objects), and a function  $f: M \to N$  (on morphisms). The latter has to preserve composition (f(mn) = f(m)f(n)) and identities (f(1) = 1).
  - (c) Functors  $G \to H$  by definition consist of a function  $f: \operatorname{Vertices}(G) \to \operatorname{Vertices}(H)$  (on objects), and a function  $g: \operatorname{Edges}(G) \to \operatorname{Paths}(H)$ . The latter induces a function  $\operatorname{Paths}(G) \to \operatorname{Paths}(H)$  that respects associativity of composition and identities by definition of composition and identities in the category G.
- **Exercise 0.5.** (a) Composition of monotone functions is monotone, and the identity is a monotone function.
  - (b) Composition of homomorphisms is a homomorphism, and the identity is a homomorphism.
- **Exercise 0.6.** (a) Isomorphisms are clearly bijections. Conversely, suppose  $f: A \to B$  is a bijection. Then there exists a function  $f^{-1}: B \to A$  with  $f(f^{-1}(b)) = b$  and  $f(f^{-1}(a)) = a$ . So f is an isomorphism.
  - (b) Isomorphisms are clearly bijective morphisms. Conversely, suppose  $f: M \to N$  is a bijective morphism. Then there exists a function  $f^{-1}: N \to M$  that inverts it. We have to show that  $f^{-1}$  is a homomorphism. Clearly  $f^{-1}(1) = f^{-1}(f(f^{-1}(1))) = f^{-1}(f(1)) = 1$ . Similarly  $f^{-1}(xy) = f^{-1}(f(f^{-1}(x))f(f^{-1}(y))) = f^{-1}(f(f^{-1}(x))f^{-1}(y)) = f^{-1}(x)f^{-1}(y)$ .
  - (c) Let P be the partially ordered set  $\{0, 1\}$  where 0 and 1 are incomparable:  $0 \leq_P 1$  nor  $1 \leq_P 0$ . Let Q be the set  $\{0, 1\}$  partially ordered by  $0 \leq_Q 1$  (but not  $1 \leq_Q 0$ ). Let  $f: P \rightarrow Q$  be the function f(0) = 0 and f(1) = 1. Then f is bijective and monotone. Its inverse would have to be the set-theoretic function  $Q \rightarrow P$  given by  $0 \mapsto 0$  and  $1 \mapsto 1$ , but that function is not monotone.
- **Exercise 0.7.** (a) True: the functor that sends a set A to itself, and a relation  $R \subseteq A \times B$  to  $\{(b,a) \mid (a,b) \in R\} \subseteq B \times A$ , is its own inverse.
  - (b) False: if there were an isomorphism, then  $\mathbf{Set}(A, B) \simeq \mathbf{Set}(B, A)$  for any sets A, B. But for e.g.

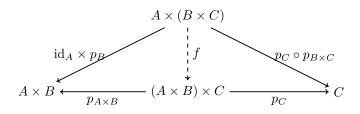
 $A = \{*\}$  and  $B = \{0, 1\}$  these two (hom)sets have different cardinality.

- (c) True: the assignment on objects that sends  $U \in P(X)$  to its complement  $X \setminus U \in P(X)$  is functorial, and its own inverse.
- **Exercise 0.8.** (a) A product of x and y is by definition an object  $x \wedge y$  such that  $x \ge x \wedge y \le y$ . It has to satisfy the universal property: if there is another object z with  $x \ge z \le y$ , then there is a (unique) morphism  $z \le x \wedge y$ .
  - (b) Reverse all the inequality signs.

**Exercise 0.9.** The universal property of  $A \times B$  provides a morphism that we'll call  $id_A \times p_B$ :



The universal property of  $(A \times B) \times C$  now provides a morphism  $f \colon A \times (B \times C) \to (A \times B) \times C$ :



Similarly we find a morphism  $g: (A \times B) \times C \to A \times (B \times C)$ .

Now  $p_A \circ (g \circ f) = p_A \circ id_{A \times (B \times C)}$  and  $p_{B \times C} \circ (g \circ f) = p_{B \times C} \circ id_{A \times (B \times C)}$ . But the universal property of  $A \times (B \times C)$  says there is only one morphisms that can satisfy this, so we must have  $g \circ f = id$ . Similarly  $f \circ q = id$ .