Categories and Quantum Informatics exercise sheet 1 answers:
Categorical semantics

Exercise 0.1. Composition arises from transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$. This is automatically associative. Identities arise from reflexivity: $x \leq x$. (We don’t actually need anti-symmetry, pre-orders also induce categories this way.)

Exercise 0.2. Associativity of the composition of the category is precisely associativity of the monoid multiplication.

Note: pre-orders and monoids are two ‘extreme’ types of categories. Pre-orders have lots of objects and as few morphisms as possible. Monoids have as few objects as possible and lots of morphisms. In a sense any category is a mixture of these two extremes.

Exercise 0.3. Concatenating paths is associative. Identities arise from paths $v \mapsto v$ of length 0.

Exercise 0.4. (a) A functor $P \to Q$ by definition consists of a function $f : P \to Q$ (on objects) that maps morphisms to morphisms. This means precisely that if $x \leq y$ is a morphism in $P$, then there must be a morphism $f(x) \leq f(y)$ in $Q$.

(b) A functor $M \to N$ by definition consists of a function $\{\ast\} \to \{\ast\}$ (on objects), and a function $f : M \to N$ (on morphisms). The latter has to preserve composition ($f(mn) = f(m)f(n)$) and identities ($f(1) = 1$).

(c) Functors $G \to H$ by definition consist of a function $f : \text{Vert}(G) \to \text{Vert}(H)$ (on objects), and a function $g : \text{Edge}(G) \to \text{Path}(H)$. The latter induces a function $\text{Path}(G) \to \text{Path}(H)$ that respects associativity of composition and identities by definition of composition and identities in the category $G$.

Exercise 0.5. (a) Composition of monotone functions is monotone, and the identity is a monotone function.

(b) Composition of homomorphisms is a homomorphism, and the identity is a homomorphism.

Exercise 0.6. (a) Isomorphisms are clearly bijections. Conversely, suppose $f : A \to B$ is a bijection. Then there exists a function $f^{-1} : B \to A$ with $f(f^{-1}(b)) = b$ and $f(f^{-1}(a)) = a$. So $f$ is an isomorphism.

(b) Isomorphisms are clearly bijective morphisms. Conversely, suppose $f : M \to N$ is a bijective morphism. Then there exists a function $f^{-1} : N \to M$ that inverts it. We have to show that $f^{-1}$ is a homomorphism. Clearly $f^{-1}(1) = f^{-1}(f(f^{-1}(1))) = f^{-1}(f(1)) = 1$. Similarly $f^{-1}(xy) = f^{-1}(f(f^{-1}(x))f(f^{-1}(y))) = f^{-1}(f(f^{-1}(x)f^{-1}(y))) = f^{-1}(x)f^{-1}(y)$.

(c) Let $P$ be the partially ordered set $\{0, 1\}$ where 0 and 1 are incomparable: $0 \not\leq_P 1$ nor $1 \not\leq_P 0$. Let $Q$ be the set $\{0, 1\}$ partially ordered by $0 \leq_Q 1$ (but not $1 \leq_Q 0$). Let $f : P \to Q$ be the function $f(0) = 0$ and $f(1) = 1$. Then $f$ is bijective and monotone. Its inverse would have to be the set-theoretic function $Q \to P$ given by $0 \mapsto 0$ and $1 \mapsto 1$, but that function is not monotone.

Exercise 0.7. (a) True: the functor that sends a set $A$ to itself, and a relation $R \subseteq A \times B$ to $\{(b, a) \mid (a, b) \in R\} \subseteq B \times A$, is its own inverse.

(b) False: if there were an isomorphism, then $\text{Set}(A, B) \simeq \text{Set}(B, A)$ for any sets $A, B$. But for e.g.
A = \{*\} and B = \{0, 1\} these two (hom)sets have different cardinality.

(c) True: the assignment on objects that sends U \in P(X) to its complement X \setminus U \in P(X) is functorial, and its own inverse.

Exercise 0.8.  (a) A product of x and y is by definition an object x \land y such that x \geq x \land y \leq y. It has to satisfy the universal property: if there is another object z with x \geq z \leq y, then there is a (unique) morphism z \leq x \land y.

(b) Reverse all the inequality signs.

Exercise 0.9. The universal property of A \times B provides a morphism that we'll call \text{id}_A \times p_B:

\begin{tikzpicture}
  
  \node (A) at (0,0) {A \times (B \times C)};
  \node (B) at (5,0) {B};
  \node (C) at (0,-5) {A \times B};
  \node (D) at (5,-5) {B \times C};
  \node (E) at (5,-10) {A \times B \times C};

  \draw[->] (A) -- (B) node[midway,above] {\text{id}_A \times p_B};
  \draw[->] (A) -- (C) node[midway,above] {p_A \times p_B};
  \draw[->] (A) -- (D) node[midway,above] {p_B \circ p_B \times C};
  \draw[->] (C) -- (B) node[midway,above] {p_B};

  \draw[->] (E) -- (A) node[midway,below] {p_A};
  \draw[->] (E) -- (C) node[midway,below] {p_B \circ p_B \times C};
  \draw[->] (E) -- (D) node[midway,below] {p_B \times C};

\end{tikzpicture}

The universal property of (A \times B) \times C now provides a morphism f: A \times (B \times C) \to (A \times B) \times C:

\begin{tikzpicture}
  
  \node (A) at (0,0) {A \times (B \times C)};
  \node (B) at (5,0) {C};
  \node (C) at (0,-5) {A \times B \times C};
  \node (D) at (5,-5) {A \times B \times C};

  \draw[->] (A) -- (B) node[midway,above] {p_C \circ p_B \times C};
  \draw[->] (A) -- (C) node[midway,above] {\text{id}_A \times p_B \circ p_B \times C};
  \draw[->] (C) -- (B) node[midway,above] {\text{id}_A \times p_B \circ p_B \times C};

\end{tikzpicture}

Similarly we find a morphism g: (A \times B) \times C \to A \times (B \times C).

Now p_A \circ (g \circ f) = p_A \circ \text{id}_{A \times (B \times C)} and p_B \times C \circ (g \circ f) = p_B \times C \circ \text{id}_{A \times (B \times C)}. But the universal property of A \times (B \times C) says there is only one morphisms that can satisfy this, so we must have g \circ f = \text{id}. Similarly f \circ g = \text{id}.