Computer Programming: Skills & Concepts (CP)
Recursion (including `mergesort`)

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Today's lecture

- Recursion: functions that call themselves.
- Examples: `factorial` and `fibonacci`.
- The `long` integer type.
- MergeSort (recursive version).
- Allocating memory dynamically with `calloc`.

Computing Factorial

Task: write a function that computes factorial

$$n! = \begin{cases} n \times (n-1)! & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

Factorial with `for` loop

```
long factorial(int n) {
  long fact = 1;
  int i;
  for(i=1; i<=n; i++ ) {
    fact = fact * i;
  }
  return fact;
}
```

We use `long` (meaning 'long int') rather than `int`, because as `n` increases, `factorial(n)` can get very large - for example 13! is too large for an `int` variable (on DICE).

On DICE, `long` uses 64 bits - can store values up to $\pm 9.22337204 \times 10^{18}$.
System dependent: On old machines, `long` might be 32 bits only.

Factorial with recursion

```
long factorial(int n) {
  if (n<=1) {
    return 1;
  }
  return n * factorial(n-1);
}
```

The function `factorial` calls itself!
A function is a 'black box' to which you give arguments, and it returns a result. Recursive calls are no different from any other call!
`factorial(n)` needs to know what $(n-1)!$ is, so it just calls the black box `factorial(n-1)`.
`factorial(1)` doesn't need to call anything.

Execution of recursion

If you look at how the sequence of execution goes, it's like this:

```
factorial(5)
    return 5 * factorial(4);
               return 4 * factorial(3);
                          return 3 * factorial(2);
                                     return 2 * factorial(1);
                                                return 1;
                                     return 2 * 1;
                          return 3 * 2
               return 4 * 6
    return 5 * 24
120
```

but just think in terms of the black box.

Fibonacci numbers

The Fibonacci numbers are the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

$$F(n) = \begin{cases} F(n-1) + F(n-2) & \text{if } n > 1 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$\frac{F(n+1)}{F(n)}$ converges to the golden ratio $\frac{1+\sqrt{5}}{2} = 1.618034$.

Recursive computation of Fibonacci numbers

```
long fibonacci(int n) {
  if (n==0)
    return 0;
  if (n==1)
    return 1;
  return fibonacci(n-1) + fibonacci(n-2);
}
```

- How many function calls does it roughly take to compute `fibonacci(10)` or `fibonacci(100)`?
- Running time?
  - Is this faster/slower than the `for` and `while` versions from lecture 6?
  - Interesting comparison using `clock()` :-).
    more interesting than comparing linear against binary search.

Merge

Idea:

Suppose we have two arrays `a`, `b` of length `n`; and `m` respectively, and that these arrays **are already sorted**. Then the merge of `a` and `b` is the sorted array of length `n+m` we get by walking through both arrays jointly, taking the smallest item at each step.

a		b		c	
2 7 18	↑ i=0	4 5 6	↑ j=0		↑ k=0
2 7 18	⇑ i=0	4 5 6	↑ j=0	2	⇑ k=0
2 7 18	↑ i=1	4 5 6	↑ j=0	2	↑ k=1
2 7 18	↑ i=1	4 5 6	⇑ j=0	2 4	⇑ k=1

a		b		c	
2 7 18	↑ i=1	4 5 6	↑ j=1	2 4	↑ k=2
2 7 18	↑ i=1	4 5 6	⇑ j=1	2 4 5	⇑ k=2
2 7 18	↑ i=1	4 5 6	↑ j=2	2 4 5	↑ k=3
2 7 18	↑ i=1	4 5 6	⇑ j=2	2 4 5 6	⇑ k=3
2 7 18	↑ i=1	4 5 6	↑ j=3	2 4 5 6	↑ k=4
2 7 18	⇑ i=1	4 5 6	↑ j=3	2 4 5 6 7	⇑ k=4
2 7 18	↑ i=2	4 5 6	↑ j=3	2 4 5 6 7	↑ k=5
2 7 18	⇑ i=2	4 5 6	↑ j=3	2 4 5 6 7 18	⇑ k=5
2 7 18	↑ i=3	4 5 6	↑ j=3	2 4 5 6 7 18	↑ k=6

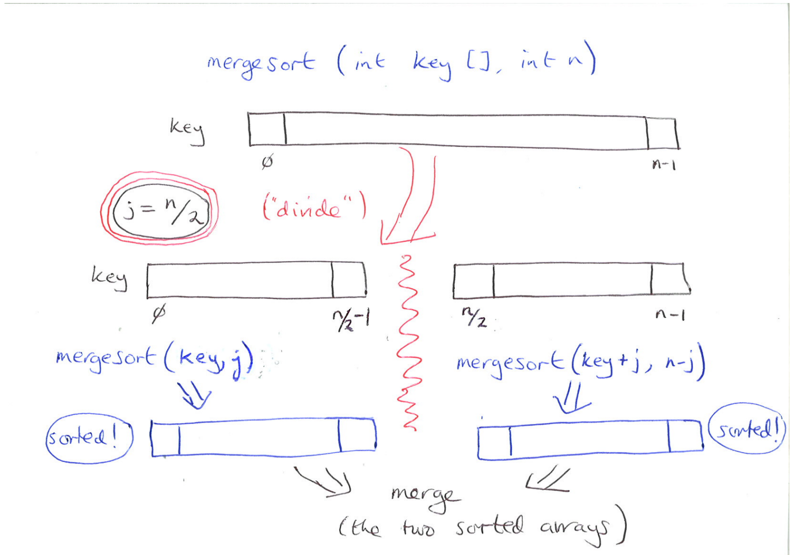
merge

```c
void merge(int a[], int b[], int c[], int m, int n) {
  int i=0, j=0, k=0;
  while (i < m || j < n) {
    /* In the following condition, if j >= n, C knows the
       condition is true, and it *doesn't* check the rest,
       so it doesn't matter that b[j] is off the end of the
       array. So the order of these is important */
    if (j >= n || (i < m && a[i] <= b[j])) {
      /* either run out of b, or the smallest elt is in a */
      c[k++] = a[i++];
    } else {
      /* either run out of a, or the smallest elt is in b */
      c[k++] = b[j++];
    }
  }
}
```


sorting using merge

- `merge` can create an *overall sort* from two smaller arrays which were individually sorted.
- Could we use `merge` as a helper function to perform the task of sorting a given array (*where the initial arrays is not at all sorted*)?
  - Divide-and-Conquer is the process of solving a big problem, by utilizing the solutions to smaller versions of that problem.
  - For sorting, we could divide our (unsorted) input array in two pieces, then sort those two smaller subarrays individually (recursion) and finally get the overall sort using merge.

sorting using merge



MergeSort through recursion

- Typical implementation of MergeSort is *recursive*:
  - *The sort of the array* `key` *is the result of sorting each half of* `key` *and then* `merge`*-ing those two sorted subarrays*
  - `merge` function takes two *sorted* arrays and creates the 'merge' of those arrays
- C issues:
  We will need to create a 'scratch array' to pass to `merge` (as the `c` parameter) to write the result of 'merging' the two smaller (already sorted) subarrays.
  - 'scratch' array needs same length as input array.
  - Need to *dynamically* allocate space.
  - In C, use `calloc` to get space of a 'dynamic' (not fixed) amount.
    `void *calloc(size_t num, size_t size)`
  - Returns a '*pointer to void*' ... just means a pointer/address of *no particular type.*
    The pointer will be `NULL` if there was not enough available space.

recursive mergesort

```
int mergesort(int key[], int n){
  int j, *w;
  if (n <= 1) { return 1; } /* base case, sorted */
  w = calloc(n, sizeof(int));    /* space for temporary array */
  if (w == NULL) { return 0; } /* calloc failed */
  j = n/2;
  /* do the subcalls and check they succeed */
  if ( mergesort(key, j)
      && mergesort(key+j, n-j) ) {
    merge(key, key+j, w, j, n-j);
    for (j = 0; j < n; ++j)
    key[j] = w[j];
    free(w);    /* Free up the dynamic memory no longer in use. */
    return 1;
  } else { /* a subcall failed */
    free(w);
    return 0;
  }
}
```


Wrap-up and Reading

- Running-time of `mergesort` is proportional to $n \lg(n)$.
  Compares favourably with `BubbleSort` (n^2).
- Read more about *recursion* in Sections 5.14, 5.15 of Kelley & Pohl.
- Read more about `calloc` in Section 6.8 of Kelley & Pohl.
- Our implementation of `mergesort` is on the course webpage.

  `mergerec.c` also has a `wrt` function for printing out small arrays, and a `main` for testing/timing on arrays of various sizes.
- Kelley & Pohl have a 'bottom-up' version of MergeSort for *array lengths* a power-of-2.
  - More troublesome/fiddly than the recursive version
  - Can adapt their 'bottom-up' version of MergeSort to work for general array lengths. *If you want a challenge try this*
    (but test it rigorously)