This lecture:
- Parallelisation for fork/join
- Mapping parallelism to shared memory multi-processors
- Loop distribution and fusion
- Data Partitioning and SPMD parallelism
- Communication, synchronisation and load imbalance.
Two approaches to parallelisation

- Traditional shared memory
  Single address space
  Based on finding parallel loop iterations

- Distributed memory compilation
  Physically distributed memory uses a mixture of both
  Focus on mapping data, computation

Can show equivalence

- Implement shared memory on distributed
- Implement distributed memory on shared
Introduction
Approaches to parallelisation

Shared memory - single address space
Introduction
Approaches to parallelisation

Shared memory - probably private caches, but looks like single address space
Introduction
Approaches to parallelisation

Distributed memory - each machine has own address space
Use message passing
Loop Parallelisation

- Assume a single address space machine. Each processor sees the same set of addresses. Do not need to know physical location of memory reference.

- Control-orientated approach. Concerned with finding independent iterations of a loop. Then map or schedule these to the processor.

- Aim: find maximum amount of parallelism and minimise synchronisation.

- Secondary aim: improve load imbalance. Inter-processor communication not considered.

- Main memory just part of hierarchy - so use uni-processor approaches.
Loop Parallelisation
Fork/join

- Fork (create) threads at beginning of loop
- Thread executes one or more iterations. Depend on later scheduling policy
- Join (synchronisation/barrier) at end of loop
- Synchronisation expensive
  - Favour outer loop parallelism
  - Loop interchange
Loop Parallelisation
DOALL Implementation

<table>
<thead>
<tr>
<th>Original</th>
<th>Driver</th>
<th>Per thread</th>
</tr>
</thead>
</table>
| Do i = 1, N  
  A(i)=B(i)  
  C(i)=A(i)  
 Enddo | p=get_num_proc()  
 fork(x_sub,p)  
 join() | SUBROUTINE x_sub()  
   p = get_num_proc()  
   z = my_id()  
   ilo = N/p * (z-1) +1  
   ihi = min(N, ilo+N/p)  
   Do i = ilo, ihi  
     A(i) = B(i)  
     C(i) = A(i)  
 Enddo  
 END |

Generate $p$ independent threads of work
- Each has private local variables, $z$, $ilo$, $ihi$
- Access shared arrays $A$, $B$ and $C$
Loop Parallelisation
Using loop interchange

Original
Do i = 1, N
    Do j = 1, M
        a(i+1,j) = a(i,j)+c
    Enddo
Enddo

Interchanged
Do j = 1, M
    Do i = 1, N
        a(i+1,j) = a(i,j)+c
    Enddo
Enddo

$O(n)$ synchronisation points
Do i = 1, N
    Parallel Do j = 1, M
        a(i+1,j) = a(i,j)+c
    Enddo
Enddo

1 synchronisation point
Parallel Do j = 1, M
    Do i = 1, N
        a(i+1,j) = a(i,j)+c
    Enddo
Enddo

Interchange has reduced synchronisation overhead from $O(N)$ to 1.
Parallelisation approach

- Loop distribution eliminates carried dependences and creates opportunity for outer-loop parallelism.
- However increases number of synchronisations needed after each distributed loop.
- Maximal distribution often finds components too small for efficient parallelisation
- Solution: fuse together parallelisable loops.
Loop Fusion

Fusion illegal if changes the dependence direction

**Two loops - same bounds**

Do i = 1, N  
  a(i) = b(i) + c  
Enddo  
Do i = 1, N  
  d(i) = a(i) + e  
Enddo

**Fused**

Do i = 1, N  
  a(i) = b(i) + c  
  d(i) = a(i) + e  
Enddo

Profitability: Parallel and sequential loops should not generally be merged
Loop Fusion

Fusion illegal if changes the dependence direction

Two loops - same bounds
Do i = 1, N
  a(i) = b(i) + c
Enddo
Do i = 1, N
  d(i) = a(i+1) + e
Enddo

Fused
Do i = 1, N
  a(i) = b(i) + c
  d(i) = a(i+1) + e
Enddo

Take care that fusing does not prevent parallelisation
Data Parallelism

- Alternative approach where we focus on mapping data rather than control flow to the machine
- Data is partitioned/distributed across the processors of the machine
- The computation is then mapped to follow the data - typically such that work writes to local data. Local write/owner computes rule.
- All of this is based on the SPMD computational model. Each processor runs one thread executing the same program, operating on the different data
- This means that loop bounds change from processor to processor.
Data Parallelism
Mapping

- Placement of work and data on processors. Assume parallelism found in a previous stage.
- Typically program parallelism $O(n)$ is much greater than machine parallelism $O(p)$, $n \gg p$.
- We have many options as to how to map a parallel program.
- Key issue: What is the best mapping that achieves $O(p)$ parallelism but minimises cost.
- Costs include communication, load imbalance and synchronisation.
Data Placement
Simple Fortran example

Dimension Integer a(4,8)
Do i = 1, 4
  Do j = 1, 8
    a(i, j) = i + j
  Enddo
Enddo

Note that here data and iteration spaces line up. Generally not the case.
Partitioning by columns of $a$ and hence iterator $j$ : Local writes

**Processor 1**
Dimension Integer
$a(4,1..2)$
Do $i = 1, 4$
  Do $j = 1, 2$
    $a(i,j) = i + j$
  Enddo
Enddo
...

**Processor 3**
Dimension Integer
$a(4,5..6)$
Do $i = 1, 4$
  Do $j = 5, 6$
    $a(i,j) = i + j$
  Enddo
Enddo
Partitioning by rows of a and hence iterator i: Local writes

**Processor 1**
Dimension Integer
a(1..1,1..8)
Do i = 1, 1
   Do j = 1, 8
      a(i,j) = i + j
   Enddo
Enddo
...

**Processor 3**
Dimension Integer
a(3..3,1..8)
Do i = 3, 3
   Do j = 1, 8
      a(i,j) = i + j
   Enddo
Enddo
Iteration space defined by loop bound constraints
Constraints are affine ($\vec{a} \vec{i} \leq \vec{c}$)
Matrix standard form ($A\vec{i} \leq \vec{c}$)
Each constraint defines half space
Iteration space is intersection of half spaces (polytope)
Iterations at integer lattice points within iteration space
  - Typically unit lattices
Array access patterns as affine functions over iteration vectors
  ($f(\vec{i}) = B\vec{i} + d$)
Linear program representation
Example

\[
\text{Do } i = 1, 16 \\
\text{Do } j = 1, 16 \\
\text{Do } k = i, 16 \\
c(i,j) = c(i,j) + a(i,k) \cdot b(j,k)
\]

Iteration constraints

\[
1 \leq i \\
1 \leq j \\
i \leq k \\
i \leq 16 \\
j \leq 16 \\
k \leq 16
\]
Linear program representation

Example

Do $i = 1, 16$
    Do $j = 1, 16$
        Do $k = i, 16$
            $c(i,j) = c(i,j) + a(i,k) \times b(j,k)$

Make into standard form

1 - $i \leq 0$
1 - $j \leq 0$
i - $k \leq 0$
i \leq 16
j \leq 16
k \leq 16
**Linear program representation**

**Example**

```
Do i = 1, 16
    Do j = 1, 16
        Do k = i, 16
            c(i,j) = c(i,j) + a(i,k)*b(j,k)
        End Do
    End Do
End Do
```

Make into standard form

\[-i \leq -1\]
\[-j \leq -1\]
\[i - k \leq 0\]
\[i \leq 16\]
\[j \leq 16\]
\[k \leq 16\]
Linear program representation
Example

Do i = 1, 16
  Do j = 1, 16
    Do k = i, 16
      c(i,j) = c(i,j) + a(i,k) * b(j,k)

Make into standard form

\[-1.0i + 0.0j + 0.0k \leq -1\]
\[0.0i - 1.0j + 0.0k \leq -1\]
\[1.0i + 0.0j - 1.0k \leq 0\]
\[1.0i + 0.0j + 0.0k \leq 16\]
\[0.0i + 1.0j + 0.0k \leq 16\]
\[0.0i + 0.0j + 1.0k \leq 16\]
Linear program representation

Example

Do i = 1, 16
    Do j = 1, 16
        Do k = i, 16
            c(i,j) = c(i,j) + a(i,k)*b(j,k)
        
Make into standard form

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
i \\
j \\
k \\
\end{bmatrix}
\leq
\begin{bmatrix}
-1 \\
-1 \\
0 \\
16 \\
16 \\
16 \\
\end{bmatrix}
\]
Linear program representation

Example

Do $i = 1, 16$
   Do $j = 1, 16$
      Do $k = i, 16$
         $c(i,j) = c(i,j)$
         $+ a(i,k) * b(j,k)$
      End Do
   End Do
End Do

Make into standard form

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  i \\
j \\
k \\
\end{bmatrix}
\leq
\begin{bmatrix}
-1 \\
-1 \\
0 \\
16 \\
16 \\
16 \\
\end{bmatrix}
\]

Access matrices $U_c U_a U_b$

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}_c
\begin{bmatrix}
i \\
j \\
k \\
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}_a
\begin{bmatrix}
i \\
j \\
k \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}_b
\begin{bmatrix}
i \\
j \\
k \\
\end{bmatrix}
\]
Many transformations\(^1\) are affine functions over linear program

**Scanning** then regenerates code

Partitioning loop for different processors by adding partition constraints

\(^1\)Skew, reverse, interchange, etc
Split four processors equally along $i$

Processor 2

Do $i = 5,8$
  Do $j = 1,16$
    Do $k = i,16$
      $c(i,j) = c(i,j) + a(i,k)b(j,k)$

Determine local array bounds $\lambda_z, \upsilon_z$ for each processor $1 \leq z \leq p$.

$\lambda_1 = 1, \lambda_2 = 5, \lambda_3 = 9, \lambda_4 = 13$
$\upsilon_1 = 4, \upsilon_2 = 8, \upsilon_3 = 12, \upsilon_4 = 16$

Determine local write constraint $\lambda_z \leq U_c \leq \upsilon_z, 5 \leq i \leq 8$ and add to polytope

Works for arbitrary loop structures and accesses
Load balancing

- Load describes amount of work each processor must do
- For simple loop bodies is number of iterations assigned to each processor
- All processors wait for slowest at join point
- Want to minimise idle time at join
Load balancing
Example

```
Do i = 1, 16
  Do j = 1, 16
    Do k = i, 16
      c(i,j) = c(i,j) + a(i,k) * b(j,k)
  End Do
End Do
End Do
```

Assuming \(c, a, b\) are to be partitioned in a similar manner
How should we partition to minimise load imbalance?

- Row (along \(i\)): processor load 928, 672, 416, 160 iterations
- Column (along \(j\)): processor load 544, 544, 544, 544 iterations

Why this variation?
Partition by row (along $i$)
Load balance
Example

Partition by column (along $j$)

Partition by “invariant” iterator $j$. 
Load balance
Polytope based

- Generally straightforward to ‘read’ from polytope
- Iteration variable with zeros elsewhere in rows and columns is ‘invariant’
- Partitioning on ‘invariant’ yields balance

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Reducing Communication

We wish to partition work and data to reduce amount of communication or remote accesses

Dimension a(n,n) b(n,n)
Do i = 1, n
  Do j = 1, n
    Do k = 1, n
      a(i,j) = b(i,k)
    Enddo
  Enddo
Enddo

How should we partition to reduce communication?
Reducing communication

Each processor has rows of $a$ and $b$ allocated to it.

Look at access pattern of second processor.

Dimension $a(n,n)$ $b(n,n)$

Do $i = 1, n$
  Do $j = 1, n$
    Do $k = 1, n$
      $a(i,j) = b(i,k)$
    Enddo
  Enddo
Enddo

The columns of $a$ scheduled to $P2$ access all of $b$ $n^2 - \frac{n^2}{p}$ remote access.
Reducing communication

Each processor has rows of $a$ and $b$ allocated to it
Look at access pattern of second processor

Dimension $a(n,n)$ $b(n,n)$
Do $i = 1$, $n$
  Do $j = 1$, $n$
    Do $k = 1$, $n$
      $a(i,j) = b(i,k)$
    Enddo
  Enddo
Enddo

The rows of $a$ scheduled to P2 access corresponding rows of $b$.
0 remote accesses.
Alignment

The first index of a and b have the same subscript a(i,j), b(i,k)

They are said to be aligned on this index

Partitioning on an aligned index makes all accesses local to that array reference

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}_a, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}_b
\]

Can transform array layout to make arrays more aligned for partitioning.

Find \( A \) such that \( AU_x \) is maximally aligned with \( U_y \)

Global alignment problem
Synchronisation

- Alignment information can also be used to eliminate synchronisation.
- Early work in data parallelisation did not focus on synchronisation.
- The placement of message passing synchronous communication between source and sink would (over!) satisfy the synchronisation requirement.
- When using data parallel on new single address space machines, have to reconsider this.
- Basic idea, place a barrier synchronisation where there is a cross-processor data dependence.
Synchronisation

```
Do i = 1, 16
  a(i) = b(i)
Enddo
Do i = 1, 16
  c(i) = a(i)
Enddo

Do i = 1, 16
  a(17-i) = b(i)
Enddo
Do i = 1, 16
  c(i) = a(i)
Enddo
```

- Barrier placed between each loop. But are they necessary?
- Data that is written always local. (local write rule)
- Data that is aligned on partitioned index is local.
- No need for barriers here
Summary

- VERY brief overview of auto-parallelism
- Parallelisation for fork/join
- Mapping parallelism to shared memory multi-processors
- Data Partitioning and SPMD parallelism
- Multi-core processor are common place
- Sure to be an active area of research for years to come
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