Communication and Concurrency
Lectures 8 & 9

Colin Stirling (cps)

School of Informatics

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A third candidate: bisimulation equivalence

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- We write $E \sim F$ if E and F are bisimilar

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- $\mathtt{Ven}_1$ not bisimilar to $\mathtt{Ven}_2$

Another example

- $$\begin{aligned}
\mathtt{Sem} &\stackrel{\text{def}}{=} \mathtt{get}.\mathtt{Sem}' \\
\mathtt{Sem}' &\stackrel{\text{def}}{=} \mathtt{put}.\mathtt{Sem}
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- Show $\mathtt{Sem2}_0 \sim \mathtt{Sem} \mid \mathtt{Sem}$

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- Show $\mathtt{Sem2}_0 \sim \mathtt{Sem} \mid \mathtt{Sem}$
- The following relation is a bisimulation

$$\begin{aligned} B \quad = \{ \quad & (\mathtt{Sem2}_0, \mathtt{Sem} \mid \mathtt{Sem}), \\ & (\mathtt{Sem2}_1, \mathtt{Sem}' \mid \mathtt{Sem}), \\ & (\mathtt{Sem2}_1, \mathtt{Sem} \mid \mathtt{Sem}'), \\ & (\mathtt{Sem2}_2, \mathtt{Sem}' \mid \mathtt{Sem}') \quad \} \end{aligned}$$

Game interpretation

Board:	Transition systems of E and F.
Material:	Two (identical) pebbles initially on the states E and F.
Players:	R (refuter) and V (verifier), R and V take turns, R moves first.
R-move:	Choose any of the two pebbles Move pebble across any transition
V-move:	Choose the other pebble choose a transition having the same label move pebble across it
R wins if:	V cannot reply to his last move.
V wins if:	R cannot move or the game goes on forever. (i.e., a draw counts as a win for V).
Theorem:	R can force a win iff E and F are not bisimilar. V can force a win iff E and F are bisimilar.

Which of the following are bisimilar?

		Y/N
$a.0$	$a.a.0$	
$a.0$	$a.0 + a.0$	
$a.0$	$a.0 \mid a.0$	
$a.a.0$	$a.0 \mid a.0$	
$a.b.0$	$a.0 \mid b.0$	
$a.b.0 + b.a.0$	$a.0 \mid b.0$	
$a.\overline{a}.0 + \overline{a}.a.0$	$a.0 \mid \overline{a}.0$	
$a.\overline{a}.0 + \overline{a}.a.0 + \tau.0$	$a.0 \mid \overline{a}.0$	
$\tau.0$	$(a.0 \mid \overline{a}.0)\backslash a$	

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$a.a.0$	$a.0 \mid a.0$	Y
$a.b.0$	$a.0 \mid b.0$	N
$a.b.0 + b.a.0$	$a.0 \mid b.0$	Y
$a.\bar{a}.0 + \bar{a}.a.0$	$a.0 \mid \bar{a}.0$	N
$a.\bar{a}.0 + \bar{a}.a.0 + \tau.0$	$a.0 \mid \bar{a}.0$	Y
$\tau.0$	$(a.0 \mid \bar{a}.0)\backslash a$	Y

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We deal first with conditions $1 - 4$

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  Proof: Since $E \sim F$, $(E, F) \in B_1$ for some bisimulation B_1. Since $F \sim G$, $(F, G) \in B_2$ for some bisimulation B_2. So $(E, G) \in B_1 \circ B_2$. We show that $B_1 \circ B_2$ is a bisimulation.

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Proposition: If $E \sim F$, then for any process G, for any set of actions K, for any action a and for any renaming function f,

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1. $a.E \sim a.F$
2. $E + G \sim F + G$
3. $E \mid G \sim F \mid G$
4. $E[f] \sim F[f]$
5. $E \backslash K \sim F \backslash K$

Proof

Assume $E \sim F$. So there is a bisimulation C with $(E, F) \in C$

1. We show that for an a, $a.E \sim a.F$
   Let $B = \{(a.E, a.F)\} \cup C$: clearly, B is a bisimulation
2. We show that for any G, $E + G \sim F + G$
   Let $B = \{E + G, F + G\} \cup C \cup I$ where I is the identity relation: clearly B is a bisimulation
3. See next slide
4. We show that for any f, $E[f] \sim F[f]$.
   Let $B = \{(G[f], H[f]) : (G, H) \in C\}$: clearly, B is a bisimulation
5. We show that for any K, $E \backslash K \sim F \backslash K$.
   Let $B = \{(G \backslash K, H \backslash K) : (G, H) \in C\}$: clearly, B is a bisimulation

Proof of case 3: if $E \sim F$ then $E \mid G \sim F \mid G$

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- $E \xrightarrow{a} E'$ and $G = G'$. Because $E \sim F$, we know that $F \xrightarrow{a} F'$ and $E' \sim F'$ for some F'. Therefore $F \mid G \xrightarrow{a} F' \mid G$, and so $((E' \mid G), (F' \mid G)) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.
- $a = \tau$ and $E \xrightarrow{b} E'$ and $G \xrightarrow{\overline{b}} G'$. $F \xrightarrow{b} F'$ for some F' such that $E' \sim F'$, so $F \mid G \xrightarrow{\tau} F' \mid G'$, and therefore $((E' \mid G'), (F' \mid G')) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.
- $a = \tau$ and $E \xrightarrow{b} E'$ and $G \xrightarrow{\overline{b}} G'$. $F \xrightarrow{b} F'$ for some F' such that $E' \sim F'$, so $F \mid G \xrightarrow{\tau} F' \mid G'$, and therefore $((E' \mid G'), (F' \mid G')) \in B$.

Symmetrically for a transition $F \mid G \xrightarrow{a} F' \mid G'$.

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- Basis: $\Phi = \mathtt{tt}$ or $\Phi = \mathtt{ff}$. Clear.
- Step: We consider only the case $\Phi = [K]\Psi$. By symmetry, it suffices to show that $G \models [K]\Psi$ implies $H \models [K]\Psi$.
  Assume $G \models [K]\Psi$. For any G' such that $G \xrightarrow{a} G'$ and $a \in K$, it follows that $G' \models \Psi$.
  Let $H \xrightarrow{a} H'$ (with $a \in K$). Since $G \sim H$, there is a G' such that $G \xrightarrow{a} G'$ and $G' \sim H'$. By the induction hypothesis $H' \models \Psi$, and therefore $H \models \Phi$.

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- E is image-finite if all processes reachable from it are immediately image-finite.

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  Need to show $H \xrightarrow{a} H_i$ and $G' \equiv_{\mathrm{HM}} H_i$
- Because $G \models \langle a \rangle \mathtt{tt}$ and $G \equiv_{\mathrm{HM}} H$, $H \models \langle a \rangle \mathtt{tt}$
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- Let $\Psi = \Phi_1 \wedge \ldots \wedge \Phi_n$.
  $G \models \langle a \rangle \Psi$ but $H \not\models \langle a \rangle \Psi$ because each H_i fails to have property Ψ. Contradicts $G \equiv_{\mathrm{HM}} H$.

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- Case $H \xrightarrow{a} H'$ is symmetric.

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- Proof: Because CTL$^-$ contains modal logic.