Communication and Concurrency
Lectures 8 & 9

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- We write $E \sim F$ if E and F are bisimilar

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- $B_1 = \{(\texttt{Cl}, \texttt{Cl}_2)\}$ is not a bisimulation
- $B_2 = \{(\texttt{Cl}, \texttt{Cl}_2), (\texttt{Cl}, \texttt{tick.Cl}_2)\}$ is a bisimulation.
- $$\begin{aligned} \texttt{Ven}_1 &\stackrel{\text{def}}{=} \texttt{1p.1p.(tea.Ven}_1 + \texttt{coffee.Ven}_1) \\ \texttt{Ven}_2 &\stackrel{\text{def}}{=} \texttt{1p.(1p.tea.Ven}_2 + \texttt{1p.coffee.Ven}_2) \end{aligned}$$

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- $\texttt{Ven}_1$ not bisimilar to $\texttt{Ven}_2$

Another example

- $$\begin{aligned} \texttt{Sem} &\stackrel{\text{def}}{=} \texttt{get.Sem}' \\ \texttt{Sem}' &\stackrel{\text{def}}{=} \texttt{put.Sem} \end{aligned}$$
  $$\begin{aligned} \texttt{Sem2}_0 &\stackrel{\text{def}}{=} \texttt{get.Sem2}_1 \\ \texttt{Sem2}_1 &\stackrel{\text{def}}{=} \texttt{get.Sem2}_2 + \texttt{put.Sem2}_0 \\ \texttt{Sem2}_2 &\stackrel{\text{def}}{=} \texttt{put.Sem2}_1 \end{aligned}$$

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- Show $\texttt{Sem2}_0 \sim \texttt{Sem} \mid \texttt{Sem}$

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- $$\begin{aligned} \mathtt{Sem} &\stackrel{\text{def}}{=} \mathtt{get.Sem'} \\ \mathtt{Sem'} &\stackrel{\text{def}}{=} \mathtt{put.Sem} \end{aligned}$$
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- Show $\mathtt{Sem2}_0 \sim \mathtt{Sem} \mid \mathtt{Sem}$
- The following relation is a bisimulation

$$\begin{aligned} B \;=\{ \; & (\mathtt{Sem2}_0, \mathtt{Sem} \mid \mathtt{Sem}), \\ & (\mathtt{Sem2}_1, \mathtt{Sem'} \mid \mathtt{Sem}), \\ & (\mathtt{Sem2}_1, \mathtt{Sem} \mid \mathtt{Sem'}), \\ & (\mathtt{Sem2}_2, \mathtt{Sem'} \mid \mathtt{Sem'}) \quad \} \end{aligned}$$

Game interpretation

Board:	Transition systems of E and F.
Material:	Two (identical) pebbles initially on the states E and F.
Players:	R (refuter) and V (verifier), R and V take turns, R moves first.
R-move:	Choose any of the two pebbles Move pebble across any transition
V-move:	Choose the other pebble choose a transition having the same label move pebble across it
R wins if:	V cannot reply to his last move.
V wins if:	R cannot move or the game goes on forever. (i.e., a draw counts as a win for V).
Theorem:	R can force a win iff E and F are not bisimilar. V can force a win iff E and F are bisimilar.

Which of the following are bisimilar?

		Y/N
a.0	a.a.0	
a.0	a.0 + a.0	
a.0	a.0 | a.0	
a.a.0	a.0 | a.0	
a.b.0	a.0 | b.0	
a.b.0 + b.a.0	a.0 | b.0	
a.$\bar{a}$.0 + $\bar{a}$.a.0	a.0 | $\bar{a}$.0	
a.$\bar{a}$.0 + $\bar{a}$.a.0 + τ.0	a.0 | $\bar{a}$.0	
τ.0	(a.0 | $\bar{a}$.0)$\backslash a$	

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a.0	a.0 + a.0	Y
a.0	a.0 | a.0	N
a.a.0	a.0 | a.0	Y
a.b.0	a.0 | b.0	N
a.b.0 + b.a.0	a.0 | b.0	Y
a.$\bar{a}$.0 + $\bar{a}$.a.0	a.0 | $\bar{a}$.0	N
a.$\bar{a}$.0 + $\bar{a}$.a.0 + τ.0	a.0 | $\bar{a}$.0	Y
τ.0	(a.0 | $\bar{a}$.0)$\backslash a$	Y

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We deal first with conditions $1 - 4$

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  Proof: Since $E \sim F$, $(E, F) \in B_1$ for some bisimulation B_1. Since $F \sim G$, $(F, G) \in B_2$ for some bisimulation B_2. So $(E, G) \in B_1 \circ B_2$. We show that $B_1 \circ B_2$ is a bisimulation.

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Bisimilarity is a congruence

Proposition: If $E \sim F$, then for any process G, for any set of actions K, for any action a and for any renaming function f,

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2. $E + G \sim F + G$
3. $E \mid G \sim F \mid G$
4. $E[f] \sim F[f]$
5. $E \backslash K \sim F \backslash K$

Proof

Assume $E \sim F$. So there is a bisimulation C with $(E, F) \in C$

1. We show that for an a, $a.E \sim a.F$
   Let $B = \{(a.E, a.F)\} \cup C$: clearly, B is a bisimulation
2. We show that for any G, $E + G \sim F + G$
   Let $B = \{E + G, F + G\} \cup C \cup I$ where I is the identity relation: clearly B is a bisimulation
3. See next slide
4. We show that for any f, $E[f] \sim F[f]$.
   Let $B = \{(G[f], H[f]) : (G, H) \in C\}$: clearly, B is a bisimulation
5. We show that for any K, $E\backslash K \sim F\backslash K$.
   Let $B = \{(G\backslash K, H\backslash K) : (G, H) \in C\}$: clearly, B is a bisimulation

Proof of case 3: if $E \sim F$ then $E \mid G \sim F \mid G$

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- $E \xrightarrow{a} E'$ and $G = G'$. Because $E \sim F$, we know that $F \xrightarrow{a} F'$ and $E' \sim F'$ for some F'. Therefore $F \mid G \xrightarrow{a} F' \mid G$, and so $((E' \mid G), (F' \mid G)) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.

- $a = \tau$ and $E \xrightarrow{b} E'$ and $G \xrightarrow{\overline{b}} G'$. $F \xrightarrow{b} F'$ for some F' such that $E' \sim F'$, so $F \mid G \xrightarrow{\tau} F' \mid G'$, and therefore $((E' \mid G'), (F' \mid G')) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.
- $a = \tau$ and $E \xrightarrow{b} E'$ and $G \xrightarrow{\overline{b}} G'$. $F \xrightarrow{b} F'$ for some F' such that $E' \sim F'$, so $F \mid G \xrightarrow{\tau} F' \mid G'$, and therefore $((E' \mid G'), (F' \mid G')) \in B$.

Symmetrically for a transition $F \mid G \xrightarrow{a} F' \mid G'$.

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  For any G and H, if $G \sim H$, then $G \models \Phi$ iff $H \models \Phi$.

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- Basis: $\Phi = \mathtt{tt}$ or $\Phi = \mathtt{ff}$. Clear.
- Step: We consider only the case $\Phi = [K]\Psi$. By symmetry, it suffices to show that $G \models [K]\Psi$ implies $H \models [K]\Psi$.
  Assume $G \models [K]\Psi$. For any G' such that $G \xrightarrow{a} G'$ and $a \in K$, it follows that $G' \models \Psi$.
  Let $H \xrightarrow{a} H'$ (with $a \in K$). Since $G \sim H$, there is a G' such that $G \xrightarrow{a} G'$ and $G' \sim H'$. By the induction hypothesis $H' \models \Psi$, and therefore $H \models \Phi$.

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- E is immediately image-finite if, for each $a \in \mathsf{A}$, the set $\{F \,:\, E \xrightarrow{a} F\}$ is finite.
- E is image-finite if all processes reachable from it are immediately image-finite.

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- Assume $G \equiv_{\mathrm{HM}} H$ and $G \xrightarrow{a} G'$
  Need to show $H \xrightarrow{a} H_i$ and $G' \equiv_{\mathrm{HM}} H_i$
- Because $G \models \langle a \rangle \mathtt{tt}$ and $G \equiv_{\mathrm{HM}} H$, $H \models \langle a \rangle \mathtt{tt}$
  So $\{H' : H \xrightarrow{a} H'\} = \{H_1, \ldots, H_n\}$ is non-empty and finite by image-finiteness.

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- If $G' \not\equiv_{\mathrm{HM}} H_i$ for each $i : 1 \leq i \leq n$, there are formulas $\Phi_1, \ldots, \Phi_n$ such that $G' \models \Phi_i$ and $H_i \not\models \Phi_i$.
  (Here we use the fact that M is closed under complement.)

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- Let $\Psi = \Phi_1 \wedge \ldots \wedge \Phi_n$.
  $G \models \langle a \rangle \Psi$ but $H \not\models \langle a \rangle \Psi$ because each H_i fails to have property Ψ. Contradicts $G \equiv_{\mathrm{HM}} H$.

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- Case $H \xrightarrow{a} H'$ is symmetric.

Bisimilarity and CTL$^-$

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- Proof: Because CTL$^-$ contains modal logic.