The (strong) bisimilarity problem

• Given: two processes E and FCommunication and Concurrency Lecture 16 Colin Stirling (cps) School of Informatics 14th November 2013

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 - Check if $(E, F) \in \sim$.

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$$\begin{array}{ccc} E & \sim_{n+1} & F \\ \downarrow a & & \downarrow a \\ E' & \sim_n & F' \end{array}$$

Key result

Proposition For all $n \ge 0$,

- 1. $\sim_n \supseteq \sim$,
- 2. $\sim_n \supseteq \sim_{n+1}$, and
- 3. If $\sim_n = \sim_{n+1}$, then $\sim_n = \sim$.

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- Step: Let E ~ F. We prove E ~_{n+1} F. Let E ^a→ E' be an arbitrary transition of E

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 Since E ~ F, there is a transition F ^a→ F' of F such that E' ~ F'. By induction hypothesis, E' ~_n F'.

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 Let E ^a→ E' be an arbitrary transition of E
 Since E ~ F, there is a transition F ^a→ F' of F such that E' ~ F'. By induction hypothesis, E' ~_n F'.
 Similarly we prove that for every transition F ^a→ F' of F there is a transition E ^a→ E' of E such that E' ~_n F'.
 By definition of ~_{n+1}, we have E ~_{n+1} F

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- ► Assume $E \sim_{n+2} F$. We prove $E \sim_{n+1} F$. Let $E \xrightarrow{a} E'$ be an arbitrary transition of E.
- Since E ∼_{n+2} F, there is a transition F → F' of F such that E' ∼_{n+1} F'.
- By induction hypothesis, $E' \sim_n F'$.
- Similarly we prove that for every transition F → F' of F there is a transition E → E' of E such that E' ~_n F'. So E ~_{n+1} F.

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- ▶ To prove $\sim_n \subseteq \sim$, we show that \sim_n is a bisimulation.
- ► Let $E \sim_n F$, and let $E \xrightarrow{a} E'$ be an arbitrary transition of E. Since $\sim_n = \sim_{n+1}$, we have $E \sim_{n+1} F$, and so there is a transition $F \xrightarrow{a} F'$ of F such that $E' \sim_n F'$.

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Scheme for the computation of \sim

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- ▶ We have $\sim_n \supseteq \sim$ by (1)
- ▶ To prove $\sim_n \subseteq \sim$, we show that \sim_n is a bisimulation.
- Let E ~_n F, and let E → E' be an arbitrary transition of E. Since ~_n=~_{n+1}, we have E ~_{n+1} F, and so there is a transition F → F' of F such that E' ~_n F'.
- Similarly we prove that for every transition F → F' of F there is a transition E → E' of E such that E' ~_n F'. So ~_n is a bisimulation.

• Compute $\sim_0, \sim_1, \sim_2, \ldots$ until $\sim_i = \sim_{i+1}$.

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- Correctness: Part (3) of the Proposition.

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- Compute $\sim_0, \sim_1, \sim_2, \ldots$ until $\sim_i = \sim_{i+1}$.
- Output \sim_i .
- ▶ Correctness: Part (3) of the Proposition.
- Termination: Assume the procedure does not terminate. Then, by part (2) of the Proposition, we have an infinite chain

 $\sim_0 \supset \sim_1 \supset \sim_2 \dots$

This contradicts the finiteness of S.

Partition refinement algorithms

► Idea: think of ~ not as a set of pairs, but as a set of equivalence classes.

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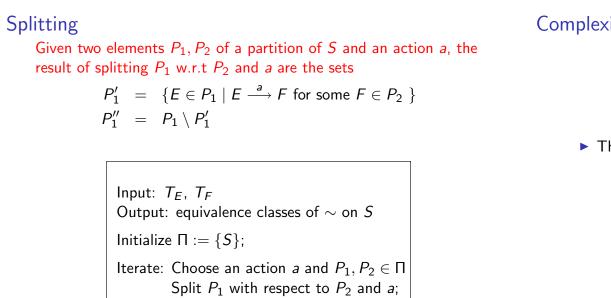
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 - all of E₁,..., E_k can do an a, and there are processes F₁,..., F_k such that E_i → F_i for every 1 ≤ i ≤ k, and moreover {F₁,..., F_k} is included in an element of the partition.
- Proof sketch: Show that the elements of a partition satisfy this property if and only if they are the equivalence classes of a bisimulation.

Show that the coarsest partition corresponds to \sim .

Splitting

Given two elements P_1 , P_2 of a partition of S and an action a, the result of splitting P_1 w.r.t P_2 and a are the sets

$$P'_1 = \{E \in P_1 \mid E \xrightarrow{a} F \text{ for some } F \in P_2 \}$$
$$P''_1 = P_1 \setminus P'_1$$



 $\Pi = (\Pi \setminus \{P_1\}) \cup \{P'_1, P''_1\};$

until a fixpoint is reached;

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- ▶ Each splitting can be performed in time $O(|S| + |\delta|)$, where $\delta = \delta_E \cup \delta_F$ (complicated).
- So the running time is $O(|S| \cdot (|S| + |\delta|))$
- Best known algorithm: $O(|\delta| \cdot log(|S|))$

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- ▶ All relations we use are subsets of $S \times S$ where S is finite.

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Compute (S, δ̂) such that for every action a (including τ) and every pair of states s, s' ∈ S, s → s' in (S, δ̂) if and only if s → s' in (S, δ).

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 - Compute (S, δ̂) such that for every action a (including τ) and every pair of states s, s' ∈ S, s → s' in (S, δ̂) if and only if s → s' in (S, δ).
 - Check if $E \sim F$ "with respect to the transition system $(S, \hat{\delta})$ ".

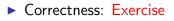
Computing $(S, \hat{\delta})$

We consider an abstract algorithm first

 $\begin{array}{l} \text{Input: } (S,\delta) \\ \text{Output: } (S,\hat{\delta}) \\ \text{Initialize } \hat{\delta} := \delta \cup \{(s,\tau,s) \mid s \in S\}; \\ \text{Iterate: For every action } a \text{ and } s,s',s'' \in S \\ \quad \text{If } (s,a,s') \in \hat{\delta} \text{ and } (s',\tau,s'') \in \hat{\delta} \text{ or} \\ \quad (s,\tau,s') \in \hat{\delta} \text{ and } (s',a,s'') \in \hat{\delta} \\ \quad \text{then add } (s,a,s'') \text{ to } \hat{\delta} \\ \text{until a fixpoint is reached;} \\ \text{return } (S,\hat{\delta}) \end{array}$

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Correctness and complexity



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- Overall time complexity: $O(|S|^5 \cdot |A|^2)$

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- Space complexity: $O(|S|^2 \cdot |A|)$

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A better algorithm

Inpi	ut: (S,δ) Output: $(S,\hat{\delta})$			
	Initialize $\hat{\delta} := \emptyset;$			
2	Initialize $\rho := \delta \cup \{(s, \tau, s) \mid s \in S\};$			
3	while $ ho eq \emptyset$ do			
4	remove $t = (s, a, s')$ from ρ ;			
5	if $t \notin \hat{\delta}$ then			
6	add t to $\hat{\delta}$;			
7	for all s'' such that $(s'', au,s)\in \hat{\delta}$			
8	$if\;(\pmb{s}'',\pmb{a},\pmb{s}')\notin\rho$			
9	then add (s'',a,s') to $ ho$;			
10	for all s'' such that $(s', au,s'')\in\hat{\delta}$			
11	if $(s, a, s'') \notin ho$			
12	then add (s,a,s'') to $ ho$;			
13	return $(S, \hat{\delta})$			
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- There is a s" such that (s, τ, s") ∈ δ and s" ⇒ s' with respect to δ. Since the shortest sequence showing s" ⇒ s' has length n − 1, by induction hypothesis (s", a, s') is eventually added to δ̂. Since any element that is moved to δ comes from ρ, (s", a, s') must be eventually added to ρ̂. By lines 7-9, (s, a, s') is also eventually added to ρ̂, and so to δ̂.

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- There is s" such that (s", τ, s') ∈ δ and s ⇒ s" with respect to δ. Analogous argument to the previous case, this time using lines lines 10-12.

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Time and space complexity

Time complexity:

- 1. Line 6 is executed $O(|S|^2 \cdot |A|)$ times. No transition can be added to $\hat{\delta}$ twice because of line 5. Since there are at most $|S| \cdot |A| \cdot |S|$ transitions, the bound follows.
- 2. Lines 8 and 11 are executed $O(|S|^3 \cdot |A|)$ times. They are executed at most once for each combination s, s', s'', a, because no element is added to $\hat{\delta}$ twice.
- 3. Line 4 is executed $O(|S|^3 \cdot |A|)$ times. By 2., $O(|S|^3 \cdot |A|)$ elements are added to ρ during the execution of the algorithm, and so $O(|S|^3 \cdot |A|)$ elements are have been removed from it after termination.
- 4. Lines 1, 2, and 13 take together $O(|S|^2 \cdot |A|)$ time.
- 5. The overall time complexity is $O(|S|^3 \cdot |A|)$.

Space complexity: since ρ and $\hat{\delta}$ do not contain duplicates, they require $O(|S|^2 \cdot |A|)$ space.

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