Communication and Concurrency
Lecture 12

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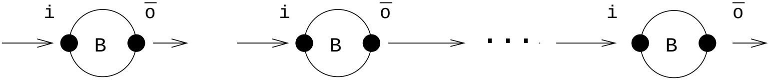

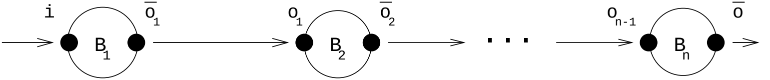

$$
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\mathtt{B}_1 & \equiv & \mathtt{B}[\mathtt{o}_1/\mathtt{o}] \\
\mathtt{B}_{j+1} & \equiv & \mathtt{B}[\mathtt{o}_j/\mathtt{i}, \mathtt{o}_{j+1}/\mathtt{o}] \quad 1 \leq j < n-1 \\
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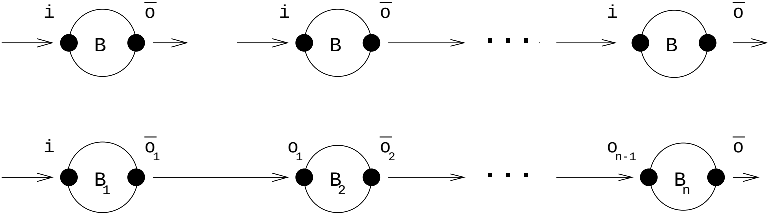

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- Redo as n Bs with $\overline{\mathtt{o}}$ linking $\mathtt{i}$: $B \frown B \frown \ldots \frown B$

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- And returns to start state

Sorting machine specification

- A multiset is a set with possibly multiple elements

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- Specification of sorter

$$\begin{array}{lcl}
\mathtt{Spec}_n & \stackrel{\text{def}}{=} & \mathtt{in}(x_1)\ldots\mathtt{in}(x_n).\mathtt{Hold}_n(\{x_1,\ldots,x_n\}) \\
\mathtt{Hold}_n(S) & \stackrel{\text{def}}{=} & \overline{\mathtt{out}}(\max(S)).\mathtt{Hold}_n(S - \{\max(S)\}) \\
 & & S \neq \emptyset \\
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- Alternatively assuming $y_1 \geq \ldots \geq y_n$

$\mathtt{Hold}_n(\{y_1,\ldots,y_n\}) \stackrel{\text{def}}{=} \overline{\mathtt{out}}(y_1)\ldots\overline{\mathtt{out}}(y_n).\overline{\mathtt{out}}(0).\mathtt{Spec}_n$

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- Use n simple cells C and a barrier cell B

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- Do it inductively
  1. Base Case: $B \approx \mathtt{Spec}_0$
  2. General Step: $\mathtt{Spec}_{n+1} \approx C \frown \mathtt{Spec}_n$
  3. Why? $\mathtt{Sorter}_{n+1} \stackrel{\mathrm{def}}{=} C \frown \mathtt{Sorter}_n$

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C'(x) &\stackrel{\text{def}}{=} \overline{\texttt{down}}(x).C + \texttt{up}(y).D(x, y) \\
D(x, y) &\stackrel{\text{def}}{=} \overline{\texttt{out}}(\max(\{x, y\})).C''(\min(\{x, y\})) \\
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- Example: $\texttt{Sorter}_3$: $C \frown C \frown C \frown B$

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  $(\mathtt{in}(z_1)\ldots\mathtt{in}(z_n).\mathtt{Hold}_n(\{z_1,\ldots,z_n\})) \approx$
- $\mathtt{in}(x_1).\tau.(C \frown (\mathtt{in}(z_2)\ldots\mathtt{in}(z_n).\mathtt{Hold}_n(\{x_1,z_2,\ldots,z_n\})))$

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- Result follows using following lemma where if S is any multiset of size k and $\{x\} \cup S = \{y_1,\ldots,y_{k+1}\}$ and $y_1 \geq \ldots \geq y_{k+1}$ then

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