Communication and Concurrency
Lecture 12

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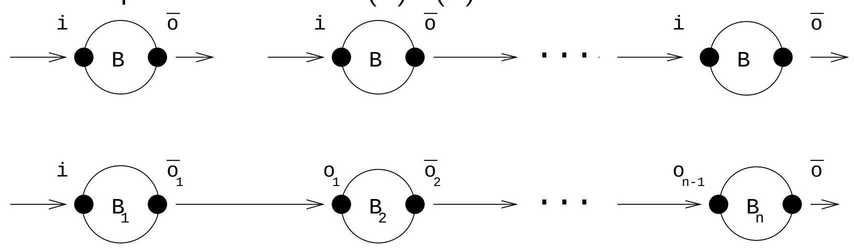

$$
\begin{array}{lll}
\mathtt{B}_1 & \equiv & \mathtt{B}[\mathtt{o}_1/\mathtt{o}] \\
\mathtt{B}_{j+1} & \equiv & \mathtt{B}[\mathtt{o}_j/\mathtt{i}, \mathtt{o}_{j+1}/\mathtt{o}] \quad 1 \leq j < n-1 \\
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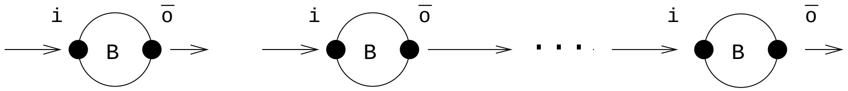

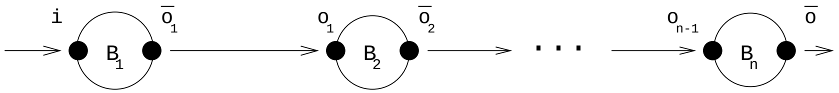

$$\begin{array}{lcll} \mathtt{B}_1 & \equiv & \mathtt{B}[\mathtt{o}_1/\mathtt{o}] & \\ \mathtt{B}_{j+1} & \equiv & \mathtt{B}[\mathtt{o}_j/\mathtt{i}, \mathtt{o}_{j+1}/\mathtt{o}] & 1 \le j < n-1 \\ \mathtt{B}_n & \equiv & \mathtt{B}[\mathtt{o}_{n-1}/\mathtt{i}] & \end{array}$$

- Redo as n Bs with $\overline{\mathtt{o}}$ linking $\mathtt{i}$: $B \frown B \frown \ldots \frown B$

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- And returns to start state

Sorting machine specification

- A multiset is a set with possibly multiple elements

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- Specification of sorter

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\mathtt{Spec}_n & \stackrel{\mathrm{def}}{=} & \mathtt{in}(x_1)\ldots\mathtt{in}(x_n).\mathtt{Hold}_n(\{x_1,\ldots,x_n\}) \\
\mathtt{Hold}_n(S) & \stackrel{\mathrm{def}}{=} & \overline{\mathtt{out}}(\max(S)).\mathtt{Hold}_n(S-\{\max(S)\}) \\
 & & S \neq \emptyset \\
\mathtt{Hold}_n(\emptyset) & \stackrel{\mathrm{def}}{=} & \overline{\mathtt{out}}(0).\mathtt{Spec}_n
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- Alternatively assuming $y_1 \geq \ldots \geq y_n$
  $\mathtt{Hold}_n(\{y_1,\ldots,y_n\}) \stackrel{\mathrm{def}}{=} \overline{\mathtt{out}}(y_1)\ldots\overline{\mathtt{out}}(y_n).\overline{\mathtt{out}}(0).\mathtt{Spec}_n$

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- Use n simple cells C and a barrier cell B

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- Do it inductively
  1. Base Case: $B \approx \mathtt{Spec}_0$
  2. General Step: $\mathtt{Spec}_{n+1} \approx C \frown \mathtt{Spec}_n$
  3. Why? $\mathtt{Sorter}_{n+1} \stackrel{\mathrm{def}}{=} C \frown \mathtt{Sorter}_n$

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C'(x) & \stackrel{\text{def}}{=} & \overline{\texttt{down}}(x).C + \texttt{up}(y).D(x,y) \\
D(x,y) & \stackrel{\text{def}}{=} & \overline{\texttt{out}}(\max(\{x,y\})).C''(\min(\{x,y\})) \\
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- Example: $\texttt{Sorter}_3$: $C \frown C \frown C \frown B$

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- $C'(x) \frown \texttt{Hold}_n(S) \approx$
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