Showing bisimilarity

To establish $E \sim F$

1. Present a candidate relation $R$ with $(E,F) \in R$
2. Prove that indeed it obeys the hereditary conditions

Example: $(A\mid B) \c c \sim C_1$

$$
\begin{align*}
A & \overset{\text{def}}{=} a.\tau.A \\
B & \overset{\text{def}}{=} c.\bar{b}.B \\
C_0 & \overset{\text{def}}{=} \bar{b}.C_1 + a.C_2 \\
C_1 & \overset{\text{def}}{=} a.C_3 \\
C_2 & \overset{\text{def}}{=} \bar{b}.C_3 \\
C_3 & \overset{\text{def}}{=} \tau.C_0
\end{align*}
$$

$R$ below is a bisimulation

$\{(((A\mid B) \c c, C_1), ((\tau.\bar{a}.\mid A\mid B) \c c, C_3), ((A\mid \bar{b}.\mid B) \c c, C_0), ((\bar{a}.\mid A\mid \bar{b}.\mid B) \c c, C_2)\}$
Showing Bisimilarity II

Some Results

A bigger example: $\text{Cnt} \sim \text{Ct}_0'$

$$
\begin{align*}
\text{Cnt} & \overset{\text{def}}{=} \up\text{(Cnt | down.0)} \\
\text{Ct}_0' & \overset{\text{def}}{=} \up\text{Ct}_i' \\
\text{Ct}_{i+1}' & \overset{\text{def}}{=} \up\text{Ct}_{i+2}' + \text{down.Ct}_i' \ i \geq 0.
\end{align*}
$$

Corollary $\sim$ is the largest bisimulation

Proposition Assume $B_i$ ($i = 1, 2, \ldots$) is a bisimulation. Then the following are bisimulations:

1. $\text{Id}
2. B_i^{-1}
3. B_1B_2
4. \bigcup\{B_i : i \geq 1\}$

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\end{align*}
$$

$$
\begin{align*}
P_0 & = \{\text{Cnt} \mid 0^j : j \geq 0\} \\
P_{i+1} & = \{E \mid 0^j \mid \text{down.0} \mid 0^k : E \in P_i \text{ and } j \geq 0 \text{ and } k \geq 0\}
\end{align*}
$$

where $F \mid 0^0 = F$ and $F \mid 0^{j+1} = F \mid 0^j \mid 0$ and brackets are dropped between parallel components.
A bigger example: \( \text{Cnt} \sim \text{Ct}_0' \)

\[
\begin{align*}
\text{Cnt} & \overset{\text{def}}{=} \text{up.(Cnt | down.0)} \\
\text{Ct}_0' & \overset{\text{def}}{=} \text{up.Ct}' \\
\text{Ct}_i' & +1 \overset{\text{def}}{=} \text{up.Ct}_i' + \text{down.Ct}_i' & i \geq 0.
\end{align*}
\]

\[
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P_0 & = \{ \text{Cnt} | 0^j : j \geq 0 \} \\
P_{i+1} & = \{ E | 0^j | \text{down.0} | 0^k : E \in P_i \text{ and } j \geq 0 \text{ and } k \geq 0 \}
\end{align*}
\]

where \( F | 0^0 = F \) and \( F | 0^i+1 = F | 0^i | 0 \) and brackets are dropped between parallel components.

\[B = \{(E,\text{Ct}_i') : i \geq 0 \text{ and } E \in P_i \}\text{ is a bisimulation}\]

More Properties I

Proposition
1. \( E + F \sim F + E \)
2. \( E + (F + G) \sim (E + F) + G \)
3. \( E + 0 \sim E \)
4. \( E + E \sim E \)

More Properties II

Proposition
1. \( (E + F)\setminus K \sim E\setminus K + F\setminus K \)
2. \( (a.E)\setminus K \sim 0 \) if \( a \in K \cup \overline{K} \)
3. \( (a.E)\setminus K \sim a.(E\setminus K) \) if \( a \notin K \cup \overline{K} \)
Expansion law

- Assume $x_i \sim \sum \{a_{ij} \cdot x_{ij} : 1 \leq j \leq n_i \}$ for $i : 1 \leq i \leq m$
- Then $x_1 | \ldots | x_m \sim \text{SUM1} + \text{SUM2}$
- SUM1 is $\sum \{a_{ij} | \tilde{x}_{21} + c_{ij} | \tilde{x}_{22} : 1 \leq i \leq m \}$
- $x_1 | x_2 \sim a_{ij} | \tilde{x}_{11} | x_2 + b_{ij} | \tilde{x}_{12} | x_2 + a_{ij} | \tilde{x}_{13} | x_2 + \tau_{ij} | \tilde{x}_{11} | x_2 + \tau_{ij} | \tilde{x}_{13} | x_2$
Expansion law

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- Then \( x_1 \mid \ldots \mid x_m \sim \text{SUM1} + \text{SUM2} \)
- \text{SUM1} is \( \sum \{ a_{ij} \cdot y_{ij} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n_i \} \)
- \text{SUM2} is \( \sum \{ \tau \cdot y_{klij} : 1 \leq k < i \leq m \text{ and } a_{kl} = \overline{a}_{ij} \} \)
- \( y_{ij} = x_1 \mid \ldots \mid x_{i-1} \mid x_j \mid x_{i+1} \mid \ldots \mid x_m \)
- \( y_{klij} = x_1 \mid \ldots \mid x_{k-1} \mid x_{kl} \mid x_{k+1} \mid \ldots \mid x_j \mid x_{i+1} \mid \ldots \mid x_m \)
- Example
  \[ x_1 \sim a \cdot x_{11} + b \cdot x_{12} + a \cdot x_{13} \]
  \[ x_2 \sim \overline{a} \cdot x_{21} + c \cdot x_{22}, \]

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\[ x_1 \mid x_2 \sim a \cdot (x_{11} \mid x_2) + b \cdot (x_{12} \mid x_2) + a \cdot (x_{13} \mid x_2) + \overline{a} \cdot (x_1 \mid x_{21}) + c \cdot (x_1 \mid x_{22}) + \tau \cdot (x_{11} \mid x_{21}) + \tau \cdot (x_{13} \mid x_{21}). \]
A binary relation $B$ between processes is a weak (or observable) bisimulation provided that, whenever $(E, F) \in B$ and $a \in O \cup \{\varepsilon\}$,

- if $E \xrightarrow{a} E'$ then $F \xrightarrow{a} F'$ for some $F'$ such that $(E', F') \in B$ and
- if $F \xrightarrow{a} F'$ then $E \xrightarrow{a} E'$ for some $E'$ such that $(E', F') \in B$

Two processes $E$ and $F$ are weak bisimulation equivalent (or weakly bisimilar) if there is a weak bisimulation relation $B$ such that $(E, F) \in B$. We write $E \approx F$ if $E$ and $F$ are weakly bisimilar.
Showing weak bisimilarity \( \approx \)

1. Present a candidate relation \( R \) with \((E,F) \in R\)
2. Prove that indeed it obeys the hereditary conditions

<table>
<thead>
<tr>
<th>(a.\tau.b.0)</th>
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Which of the following are weakly bisimilar?

Y/N

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Showing weak bisimilarity $\approx$

1. Present a candidate relation $R$ with $(E,F) \in R$
2. Prove that indeed it obeys the hereditary conditions
3. Example

$$A_0 \triangleq a.A_0 + b.A_1 + \tau.A_1$$
$$A_1 \triangleq a.A_1 + \tau.A_2$$
$$A_2 \triangleq b.A_0$$

$$B_1 \triangleq a.B_1 + \tau.B_2$$
$$B_2 \triangleq b.B_1$$

4. $A_0 \approx B_1$

$$\{(A_0,B_1),(A_1,B_1),(A_2,B_2)\}$$

is a weak bisimulation

Weak bisimulation: less redundancy

$\triangleright$ For $a \in A$ let $\hat{a}$ be $a$ if $a \neq \tau$, and let $\hat{\tau}$ be $\epsilon$.

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1. if $E \xrightarrow{a} E'$ then $F \xrightarrow{\hat{a}} F'$ for some $F'$ such that $(E', F') \in B$,
2. if $F \xrightarrow{a} F'$ then $E \xrightarrow{\hat{\tau}} E'$ for some $E'$ such that $(E', F') \in B$.

Two processes are ob equivalent, denoted by $\approx'$, if they are related by an ob bisimulation relation.

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Proposition
For \( a \in A \) let \( \hat{a} \) be \( a \) if \( a \neq \tau \), and let \( \hat{\tau} \) be \( \varepsilon \).

A binary relation \( B \) between processes is an ob bisimulation just in case whenever \((E,F) \in B\) and \( a \in A \),
1. if \( E \xrightarrow{a} E' \) then \( F \xrightarrow{\hat{a}} F' \) for some \( F' \) such that \((E',F') \in B\),
2. if \( F \xrightarrow{a} F' \) then \( E \xrightarrow{\hat{a}} E' \) for some \( E' \) such that \((E',F') \in B\).

Two processes are ob equivalent, denoted by \( \approx' \), if they are related by an ob bisimulation relation.

Proposition
1. \( B \) is a weak bisim if, and only if \( B \) is an ob bisim
2. \( \approx = \approx' \)
Properties of weak bisimulation

\[
\begin{align*}
\text{Id} & = \{(E, E)\} \\
B^{-1} & = \{ (E, F) : (F, E) \in B \} \\
B_1B_2 & = \{ (E, G) : \text{there is } F. (E, F) \in B_1 \\
& \quad \text{and } (F, G) \in B_2 \}
\end{align*}
\]

Proposition Assume \(B_i (i = 1, 2, \ldots)\) is a weak bisimulation. Then the following are weak bisimulations:

1. \(\text{Id}\)
2. \(B_i^{-1}\)
3. \(B_1B_2\)
4. \(\bigcup\{B_i : i \geq 1\}\)

Corollary \(\approx\) is the largest weak bisimulation

Proposition If \(E \sim F\) then \(E \approx F\)
Properties of weak bisimulation

\[ \text{Id} = \{(E, E)\} \]
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**Proposition** Assume \( B_i \ (i = 1, 2, \ldots) \) is a weak bisimulation. Then the following are weak bisimulations:

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**Corollary** \( \approx \) is the largest weak bisimulation

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**Corollary** \( \approx \) is the largest weak bisimulation

**Proposition** If \( E \sim F \) then \( E \approx F \)
Tau laws

1. \( a.\tau.E \approx a.E \)
2. \( E + \tau.E \approx \tau.E \)
3. \( a.(E + \tau.F) + a.F \approx a.(E + \tau.F) \)

But \( \approx \) is not a congruence with respect to the + operator. (It is a congruence w.r.t the other operators of CCS.)
Due to initial preemptive power of \( \tau \)

\( E \approx \tau.E \) but many cases \( E + F \not\approx \tau.E + F \)
\( a.0 \approx \tau.a.0 \) but \( a.0 + b.0 \not\approx \tau.a.0 + b.0 \)

\( \approx^c \) is the largest subset of \( \approx \) that is also a congruence.
But

▶ ≈ is not a congruence with respect to the + operator. (It is a congruence w.r.t the other operators of CCS.)

Due to initial preemptive power of τ

▶ E ⊇ τ.E but many cases E + F ⊈ τ.E + F
  a.0 ⊇ τ.a.0 but a.0 + b.0 ⊈ τ.a.0 + b.0

▶ ≈c is the largest subset of ≈ that is also a congruence.

▶ ≈ is a congruence for all the other operators of CCS.

Defining ≈c directly

E ≈c F iff

1. E ≈ F
2. if E τ−→ E′, then F τ−→ F1
   ε=⇒ F′ and E′ ≈ F′ for some F1 and F′
3. if F τ−→ F′ then E τ−→ E1
   ε=⇒ E′ and E′ ≈ F′ for some E1 and E′.
Defining $\approx^c$ directly

$E \approx^c F$ iff

1. $E \approx F$
2. if $E \xrightarrow{\tau} E'$, then $F \xrightarrow{\tau} F_1 \xrightarrow{\varepsilon} F'$ and $E' \approx F'$ for some $F_1$ and $F'$
3. if $F \xrightarrow{\tau} F'$ then $E \xrightarrow{\tau} E_1 \xrightarrow{\varepsilon} E'$ and $E' \approx F'$ for some $E_1$ and $E'$. 