Supplementary note on the Hennie-Stearns Theorem

This supplementary note supplies the missing proof of the Hennie-Stearns Theorem. Its contents are non-examinable. For the sake of convenience, in this proof only, we work with TMs having two-way infinite tapes.

Exercise 1.14. Show that a TM with two-way infinite tapes can be simulated by a TM with one-way infinite tapes, and vice versa, with no increase in time or space complexity.

Proof of the Hennie-Stearns Theorem. We follow the later proof of Pippenger and Fischer (1979).

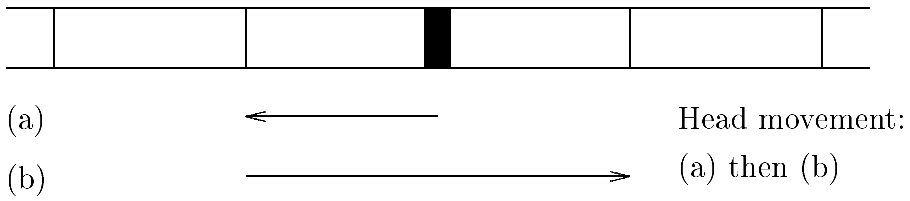
Let M_1 be the k-tape TM: we construct a two-tape TM M_2 which simulates M_1. The k tapes of M_1 are represented by one $2k$-track tape in M_2 as in the simulation of the previous section. The other tape of M_2 is used for scratch and book-keeping purposes. Call the initially scanned square of the $2k$-track tape the *home square*: the idea is to keep the simulated heads close to the home square by *shifting* the contents of the tracks. This removes the source of inefficiency in the simulation of k tapes by one.

The heart of the simulation is the following divide-and-conquer TM "procedure," which simulates n moves of M_1 in $\mathrm{O}(n \log n)$ moves of M_2 (assuming for simplicity that n is a power of 2):

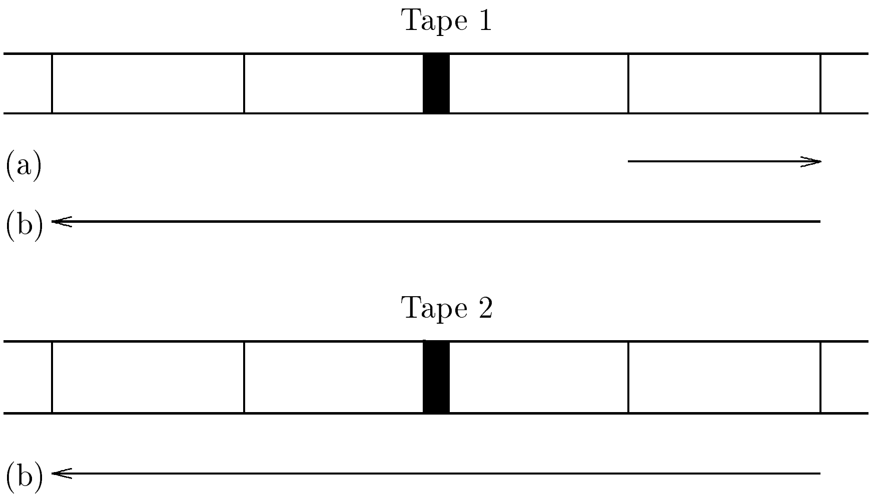
procedure *sim*(n)
if $n = 1$ **then** simulate one step of M_1, halting/accepting if
  M_1 does, and return to home square
else begin
  centre(n)
  sim$(\frac{n}{2})$
  centre(n)
  sim$(\frac{n}{2})$
  restore(n)
end

It is assumed that, on a call of *sim*(n), the positions of the simulated heads are marked on their respective tracks and are within n squares of the home square, while the head of M_2 is at the home square.

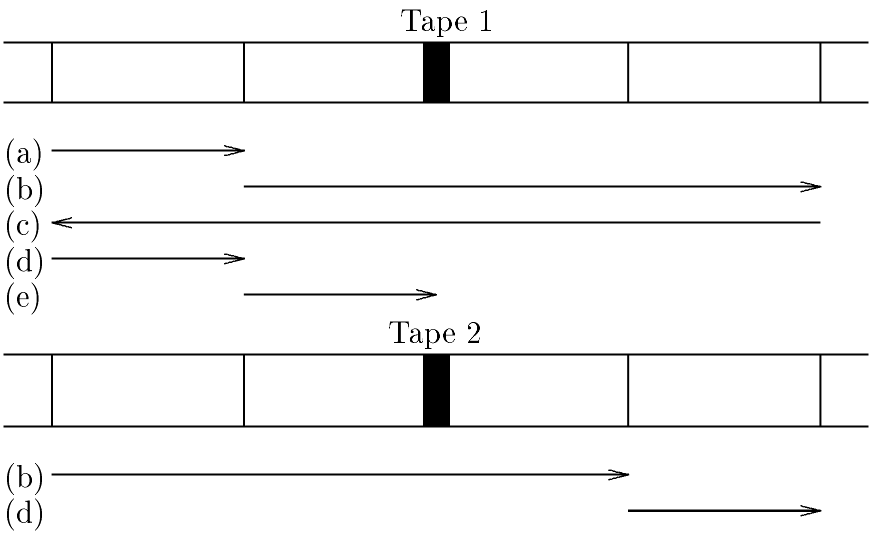
The purpose of *centre*(n) is to bring the simulated heads within $\frac{n}{2}$ squares of the home square. This is achieved, for each head in turn, by rotating the block of $4n + 1$ squares centred about the home square on the corresponding track of M_2 n positions left or right cyclically. First, *centre*(n) examines the $2n + 1$ squares about the home square to determine whether to shift left, right or not at all:



Then the block is copied to the second tape:



Finally, the block can be shifted cyclically right in O(n) time as follows:



Left cyclical shifts are implemented analogously. The above operations are performed for each of the tracks in turn.

After $\frac{n}{2}$ steps are simulated, the heads may be located as many as n squares away from the home square, so *centre*(n) is called again before the second call of *sim*($\frac{n}{2}$). Finally, the call *restore*(n) undoes the rotations performed by the two calls to *centre*(n): it shifts either left, right or not at all, and does this in $\mathrm{O}(n)$ time. (Note that the two calls to *centre*(n) cannot both cause a shift in the *same* direction, because the simulated heads cannot travel far enough in $\frac{n}{2}$ steps.) Correctness of the procedure *sim* can be established by an induction argument.

The complete simulation proceeds as follows:

$n := 1$
while not halted **do begin**
  sim(n)
  $n := 2n$
end

If $T'(n)$ is the time required for the procedure call *sim*(n), we have the familiar recurrence $T'(1) =$ constant and $T'(n) \leq 2T'(\frac{n}{2}) + \mathrm{O}(n)$, so that $T'(n) = \mathrm{O}(n \log n)$. The time taken by M_2 to simulate $T(n)$ steps of M_1 is therefore at most

$$\sum_{i=0}^{\lceil \log T(n) \rceil} \mathrm{O}(2^i \log 2^i) \;=\; \mathrm{O}(T(n) \log T(n)).$$

The TM M_2 whose construction was just sketched is not oblivious, but may easily be made so To do this, on each execution of *centre* we just make M_2 go through the head movements appropriate to both left and right shifts, writing only during the relevant shift. In similar fashion, *restore* may be implemented obliviously. □

Exercise 1.15. (a) Why can the simulation not just be the call *sim*($T(n)$)?

(b) Outline how the procedure *sim* could actually be implemented on M_2. (The key point to address is that, since the procedure is recursive, M_2 must maintain a stack of activation records: where is this to be stored?)