Second order logic

REACHABILITY asserts: for some $r \geq 0$, graph has vertices $x_1, x_2, \ldots, x_r$ such that $E(x_1, x_2)$ and $E(x_2, x_3)$ and ... and $E(x_{r-1}, x_r)$.

Problem: r not fixed so have no way of writing requirement as one formula of first order logic.

Underlying relation: REACHABILITY is captured by a binary relation on vertices.

REACHABILITY in detail: can go from vertex x to vertex y if and only if

1. there is a linear ordering of (some of) the vertices of the graph,

2. any two *consecutive* vertices in the order are joined by an edge in the graph (going from the smaller vertex to the larger one),

3. the first vertex is x, the last is y *and* x, y are in the order.

Denote linear order whose existence is to be asserted, by a binary relation symbol L.

1. L is a linear order on a subset of the vertices:

$$\begin{aligned}\psi_1 = \forall u\,\neg L(u,u)\\ \wedge\,\forall u \forall v(L(u,v) \Rightarrow \neg L(v,u))\\ \wedge\,\forall u \forall v \forall w(L(u,v) \wedge L(v,w) \Rightarrow L(u,w)).\end{aligned}$$

2. Any two consecutive vertices in the order must be joined by an edge in G:

$$\begin{aligned}\psi_2 =& \forall u \forall v((L(u,v) \wedge \forall w(\neg L(u,w) \vee \neg L(w,v)))\\ &\Rightarrow E(u,v)).\end{aligned}$$

3. The first vertex in the order is x and the last is y *and* x, y are in the order:

$$\psi_3 = (\forall u\,\neg L(u,x)) \wedge (\forall v\,\neg L(y,v)) \wedge L(x,y).$$

Tempting to take

$$\phi(x,y) = \exists L(\psi_1 \wedge \psi_2 \wedge \psi_3)$$

but this fails when $x = y$.

Easy to to overcome this, just take

$$\phi(x,y) = \exists L(x = y \vee (\psi_1 \wedge \psi_2 \wedge \psi_3)).$$

Existential second order logic

Introduce extra relation symbol R in formulae.

Allow formulae of the form

$$\exists R\,\phi$$

where ϕ looks exactly like a first order formula if R treated as an unbound variable.

Idea: R stands for a relation amongst r-tuples of elements of the structure in which formulae are interpreted (r agreed and fixed ahead of time for each formula, allowed to change it between formulae).

Note: not hard to see that a formula of the form $\exists R_1 \exists R_2 \ldots \exists R_n\,\phi$ can be expressed by one of form $\exists R\,\phi$ ('paste' the various relations into a single one).

Given $\exists R\,\phi$ have decision problem $\exists R\,\phi$-GRAPHS (similar to ϕ-GRAPHS for first order logic).

THEOREM Let $\exists R\,\phi$ be an expression of existential second order logic. Then the problem $\exists R\,\phi$-GRAPHS is in NP.

Question: Can we do better? Can we put $\exists R\,\phi$-GRAPHS in P?

With L as for REACHABILITY set:

$$\psi_4 = \forall u \forall v (L(u,v) \vee L(v,u) \vee u = v).$$

Consider:

$$\exists L(\psi_1 \wedge \psi_2 \wedge \psi_4).$$

This is Directed Hamiltonian Paths; NP-complete!

THEOREM [R. Fagin, 1974] A property of graphs is in NP if and only if it is expressible in existential second order logic.

Fagin's motivation: is existential second order logic closed under negation? i.e., given $\exists P\,\phi$ is there a formula $\exists Q\,\psi$ s.t.

$$\neg\exists P\,\phi \quad \text{is equivalent to} \quad \exists Q\,\psi?$$

Note: $\neg\exists P\,\phi$ is equivalent to $\forall P\,\neg\phi$.

Complexity version: define

$$\text{co-NP} = \{L \mid \overline{L} \text{ is in NP}\}.$$

Fagin's Theorem tells us: existential second order logic closed under negation if and only if NP = co-NP.

Note: if NP $\neq$ co-NP then P $\neq$ NP (because P = co-P).

But possible that NP = co-NP even if P $\neq$ NP.

Capturing P

Situation not quite so satisfactory. All known methods involve introduction of a concept extraneous to the logic.

One method: consider expressions of form

$$\exists R \forall x_1 \forall x_2 \ldots \forall x_n \, \phi$$

where

1. ϕ has no quantifiers,

2. ϕ is a conjunction of clauses that contain at most one un-negated instance of the relation symbol R.

Called *Horn existential second order formulae* (cf. Prolog programs).

Fairly easy argument shows: for such formulae $\exists R\, \phi$-Graphs is in P.

But: this doesn't capture all of P!

To capture P: allow a second relation symbol L which *must* be interpreted as a linear order on the vertices of any graph used to interpret ϕ.

We have no way of saying *in the logic* that L is a linear order.

THEOREM A property of graphs is in P if and only if it is expressible in Horn existential second order logic with successor.