The COLOURABILITY problem.

INSTANCE: An undirected graph G, and an integer k.
QUESTION: Is there an assignment of k colours to the vertices of G such that no two adjacent vertices receive the same colour?

THEOREM COLOURABILITY is NP-complete.

Reduction from SAT to COLOURABILITY.

Given: ϕ, instance of SAT, i.e., a Boolean formula in CNF.

Require: Graph G and integer k s.t.

$$\phi \text{ is satisfiable} \iff G \text{ is } k\text{-colourable.}$$

Formally: Let

$x_1, x_2, \ldots, x_n$:	variables in ϕ,
$C_1, C_2, \ldots, C_r$:	clauses of ϕ.

Then

$$
\begin{aligned}
V = {} & \{v_0, \ldots, v_n\} \cup \{x_1, \ldots, x_n\} \cup \{\overline{x}_1, \ldots, \overline{x}_n\} \\
& \cup \{C_1, \ldots, C_r\}, \\
E = {} & \{\{v_i, x_j\}, \{v_i, \overline{x}_j\} \mid 1 \le i, j \le n \text{ and } i \ne j\} \\
& \cup \{\{v_i, v_j\} \mid 0 \le i < j \le n\} \\
& \cup \{\{v_0, C_k\} \mid 1 \le k \le r\} \\
& \cup \{\{x_i, \overline{x}_i\} \mid 1 \le i \le n\} \\
& \cup \{\{x_i, C_k\} \mid x_i \text{ is } \textit{not} \text{ a literal in clause } C_k\} \\
& \cup \{\{\overline{x}_i, C_k\} \mid \neg x_i \text{ is } \textit{not} \text{ a literal in clause } C_k\}, \\
k = {} & n + 1.
\end{aligned}
$$

The *Integer Programming* problem, IntProg.

Instance: An integer $m \times n$ matrix A, and an integer m-vector **b**.
Question: Does there exist an integer n-vector **x** such that $A\mathbf{x} \leq \mathbf{b}$?

THEOREM IntProg is NP-complete.

Note: IntProg is in NP but tricky to show!

NP-hardness: Reduction from Sat to IntProg.

Formally: Let

$$\begin{array}{ll} x_1, x_2, \ldots, x_n\text{:} & \text{variables in } \phi, \\ C_1, C_2, \ldots, C_r\text{:} & \text{clauses of } \phi. \end{array}$$

Then for $1 \le i \le r$ and $1 \le j \le n$ define

$$\alpha_{ij} = \begin{cases} 1, & \text{if } x_j \text{ occurs in the clause } C_i; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta_{ij} = \begin{cases} 1, & \text{if } \neg x_j \text{ occurs in the clause } C_i; \\ 0, & \text{otherwise.} \end{cases}$$

Let $z_1, z_2, \ldots, z_n$ be integer variables and consider linear inequalities:

$$\begin{aligned} z_j &\ge 0, && \text{for } 1 \le j \le n; \\ z_j &\le 1, && \text{for } 1 \le j \le n; \\ \sum_{j=1}^{n} (\alpha_{ij} z_j + \beta_{ij}(1 - z_j)) &\ge 1, && \text{for } 1 \le i \le r. \end{aligned}$$

The SUBSETSUM problem.

INSTANCE: A finite set X, a positive integer 'size' $s(x)$ for each $x \in X$, and an integer 'goal' b.
QUESTION: Is there a subset $A \subseteq X$ such that $\sum_{x \in A} s(x) = b$?

THEOREM SUBSETSUM is NP-complete.

Reduction from INDSET.

Given: Undirected graph $G = (V, E)$ and positive integer k.

Require: finite set X, positive integer 'size' $s(x)$ for each $x \in X$, and integer 'goal' b s.t.

G has an independent set of size k
$\iff$ there is subset $A \subseteq X$ s.t. $\sum_{x \in A} s(x) = b$.

Formally: Let

$$\begin{array}{ll} \{v_0, \ldots, v_{n-1}\}: & \text{vertex set } V \text{ of } G, \\ \{e_0, \ldots, e_{m-1}\}: & \text{edge set } E \text{ of } G. \end{array}$$

Define

$$\begin{aligned} X &= V \cup E, \\ s(e_i) &= 10^i, \quad \text{for } 0 \le i \le m-1;, \\ s(v_i) &= 10^m + \sum_{e_j \ni v_i} 10^j, \quad \text{for } 0 \le i \le n-1, \\ b &= k \times 10^m + 10^{m-1} + 10^{m-2} + \cdots + 10^2 \\ &\quad + 10 + 1. \end{aligned}$$

THEOREM If P $\neq$ NP then there are languages in NP that are not in P and are not NP-complete.

GRAPH ISOMORPHISM

INSTANCE: Two undirected graphs $G = (V, E)$ and $G' = (V', E')$.
QUESTION: Are the graphs isomorphic?

Know: GRAPH ISOMORPHISM is in NP (obvious).

Don't Know: If it is NP-complete or in P.

Curious fact: Have *very* few natural problems of such unknown status (hence possibly in NP but not in P.)