More NP-complete problems

The *Independent Set* problem, INDSET.

INSTANCE: An undirected graph G, and an integer k.
QUESTION: Does G contain an independent set of size k?

The *Vertex Cover* problem, VC.

INSTANCE: An undirected graph G and an integer k.
QUESTION: Does G possess a vertex cover of size k?

The 3-SAT problem.

INSTANCE: A CNF Boolean formula ϕ with at most three literals per clause.
QUESTION: Is there an assignment of truth values to the variables of ϕ that makes ϕ true?

THEOREM INDSET is NP-complete.

Reduce CLIQUE to INDSET.

Given: Undirected graph $G = (V, E)$ and positive integer k; instance of CLIQUE.

Require: Production, in polynomial time, of undirected graph $G' = (\overline{V}, \overline{E})$ and positive integer k'; instance of INDSET s.t.

G has a k-clique $\iff$ G' has an independent set of size k'.

Just take

$\overline{V} = V,$

$\overline{E} = \{\{u, v\} \mid u, v \in V,\ u \neq v \text{ and } \{u, v\} \notin E\},$

$k' = k.$

THEOREM VC is NP-complete.

Reduction from INDSET; left as an exercise.

THEOREM 3-SAT is NP-complete.

Reduction from SAT.

Example:

$$x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4$$
$$\to (x_1 \vee \neg x_2 \vee y) \wedge (\neg y \vee x_3 \vee \neg x_4).$$

Generally:

$$\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_k$$
$$\to (\alpha_1 \vee \alpha_2 \vee y_1) \wedge (\neg y_1 \vee \alpha_3 \vee \cdots \vee \alpha_k)$$
$$\to (\alpha_1 \vee \alpha_2 \vee y_1) \wedge (\neg y_1 \vee \alpha_3 \vee y_2) \wedge$$
$$(\neg y_2 \vee \alpha_4 \vee \cdots \vee \alpha_k)$$
$$\to \ldots$$

Note: 2-SAT is in P, don't have any 'trick' to reduce from 3-SAT to 2-SAT!

The *Directed Hamiltonian Cycle* problem, DHC.

INSTANCE: A directed graph G.
QUESTION: Does G possess a Hamiltonian cycle, i.e., a directed closed path that visits every vertex of G precisely once?

The *Undirected Hamiltonian Cycle* problem, UHC.

INSTANCE: An undirected graph G.
QUESTION: Does G possess a Hamiltonian cycle, i.e., a closed path which visits every vertex of G precisely once?

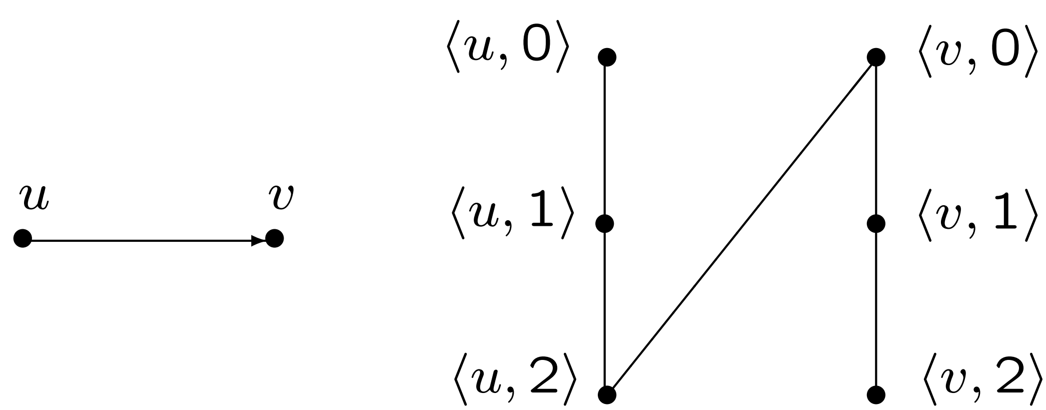
THEOREM DHC is NP-complete.

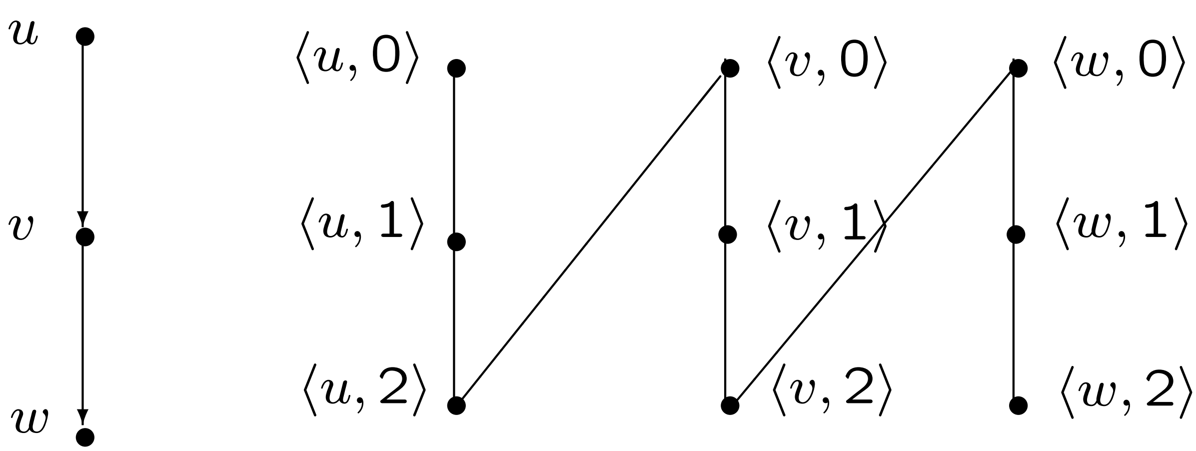
Reduction from SAT or from VC; quite complicated and amazing.

THEOREM UHC is NP-complete.

Reduction from DHC to UHC.

Gadget:

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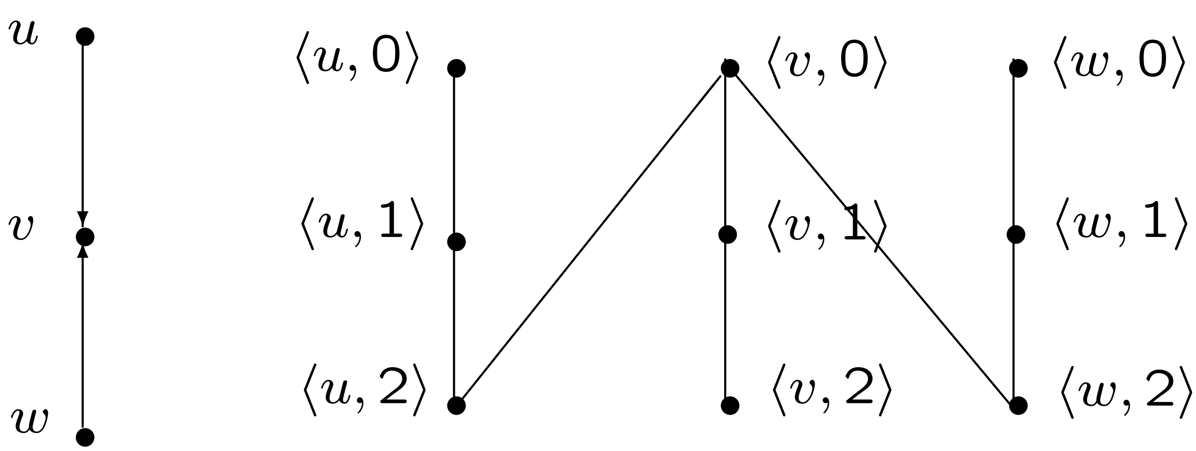

u
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$\langle u, 0\rangle$
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$\langle v, 0\rangle$
$\langle v, 1\rangle$
$\langle v, 2\rangle$
$\langle w, 0\rangle$
$\langle w, 1\rangle$
$\langle w, 2\rangle$

Formally: Given directed graph $G = (V, E)$, instance of DHC.

Define undirected graph $G' = (V', E')$ by:

$$V' = V \times \{0, 1, 2\},$$

and edge set

$$\begin{aligned} E' = \{\{\langle v, 0\rangle, \langle v, 1\rangle\} \mid v \in V\} \\ \cup \{\{\langle v, 1\rangle, \langle v, 2\rangle\} \mid v \in V\} \\ \cup \{\{\langle u, 2\rangle, \langle v, 0\rangle\} \mid (u, v) \in E\}. \end{aligned}$$

Simple argument shows

$$G \text{ has a DHC} \Longleftrightarrow G' \text{ has an UDC.}$$