Computing against the clock

Classification of languages: so far have:

- Recursive (decidable),
- Non-recursive (undecidable),
  - r.e.,
  - not r.e.

Now focus on recursive languages and ask about *cost* of solving membership problem.

Time bounded machines: M a TM with input alphabet Σ. Say that M is $T(n)$ *time bounded* (or *of time complexity* $T(n)$) if

- M halts within $T(n)$ steps on *all* inputs $x \in \Sigma^n$, i.e., all inputs of length n.

Note: $T(n)$ is just an upper bound, M could halt in fewer steps.

Say M is *polynomial time* if it is of time complexity $p(n)$ for some polynomial p.

Example: M_{palin} time bound $\frac{1}{2}(n+1)(n+2)$.

The class P: Consists of all languages over $\{0, 1\}$ that can be recognized by some polynomial time TM.

Examples:

1. Palindromes.

2. $\{0, 1\}^*$.

3. $\{\, 0^n 1^n \mid n \in \mathbb{N} \,\}$.

Observations about P:

1. P is invariant under changes of model.
2. Languages in P regarded as *tractable*.
3. Languages outside P regarded as *intractable*.
4. Really P is an idealization, or approximation, of 'practically solvable'.
5. Even so point (1) shows P is of great interest (cf. recursive languages).
6. Forced to allow all of P unless we fix a model.
7. P helps us to make precise a question about naïve search (look at this later).

Polynomial-time reductions: L_1, L_2 be languages over alphabets Σ_1 and Σ_2. A *polynomial-time reduction from* L_1 *to* L_2 is a function

$$f : \Sigma_1^* \to \Sigma_2^*$$

satisfying:

(a) $x \in L_1 \iff f(x) \in L_2, \quad$ for all $x \in \Sigma_1^*$;

(b) there is a polynomial-time Turing machine transducer that computes f.

Same definition as a reduction but now f must be computed in polynomial time.

Say L_1 *is polynomial-time reducible to* L_2, if there is a polynomial-time reduction from L_1 to L_2. Write as:

$$L_1 \leq_{\mathsf{P}} L_2.$$

Satisfying assignments and cliques:

- SAT, taken from propositional logic:

INSTANCE: A Boolean formula ϕ in conjunctive normal form (CNF).

QUESTION: Is there an assignment of truth values to the variables of ϕ that makes ϕ true?

- CLIQUE, a problem from graph theory:

INSTANCE: An undirected graph $G = (V, E)$, and an integer k.

QUESTION: Does G possess a k-clique?

Claim: SAT $\leq_P$ CLIQUE.

Given: CNF Boolean formula ϕ.

Produce: Undirected graph G and integer k s.t.

$$\phi \text{ is satisfiable} \iff G \text{ has a } k\text{-clique.}$$

Must be able to build G, k in polynomial time in the size of ϕ.

The reduction: Let

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_r,$$
$$C_i = (\alpha_{i1} \vee \alpha_{i2} \vee \cdots \vee \alpha_{i,s_i}).$$

each α_{ij} is a literal.

A pair of literals is *complementary* if it consists of the negated and un-negated forms of the *same* variable, e.g., x and $\neg x$.

Reduction maps ϕ to graph $G = (V, E)$ and integer k:

$$V = \{\alpha_{ij} \mid 1 \leq i \leq r \text{ and } 1 \leq j \leq s_i\},$$
$$E = \{\{\alpha_{ij}, \alpha_{hk}\} \mid i \neq h, \text{ and the pair } \alpha_{ij},\ \alpha_{hk} \text{ is } \textit{not} \text{ complementary}\},$$
$$k = r.$$