Computer Graphics 6 - Rasterisation

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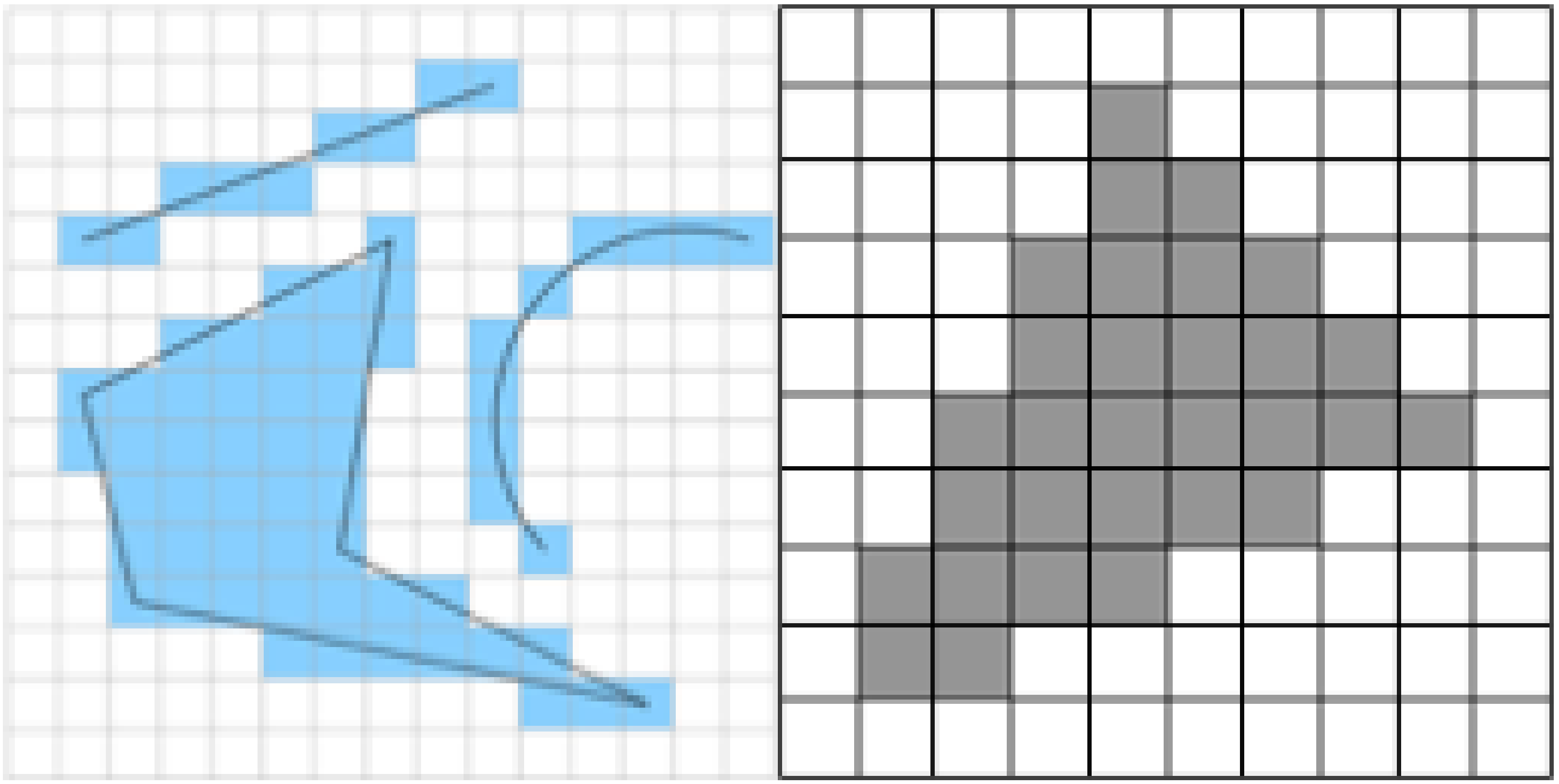
Slides courtesy of Taku Komura
www.inf.ed.ac.uk/teaching/courses/cg

Overview

- **Line rasterisation**
- Polygon rasterisation
- Mean value coordinates
- Decomposing polygons

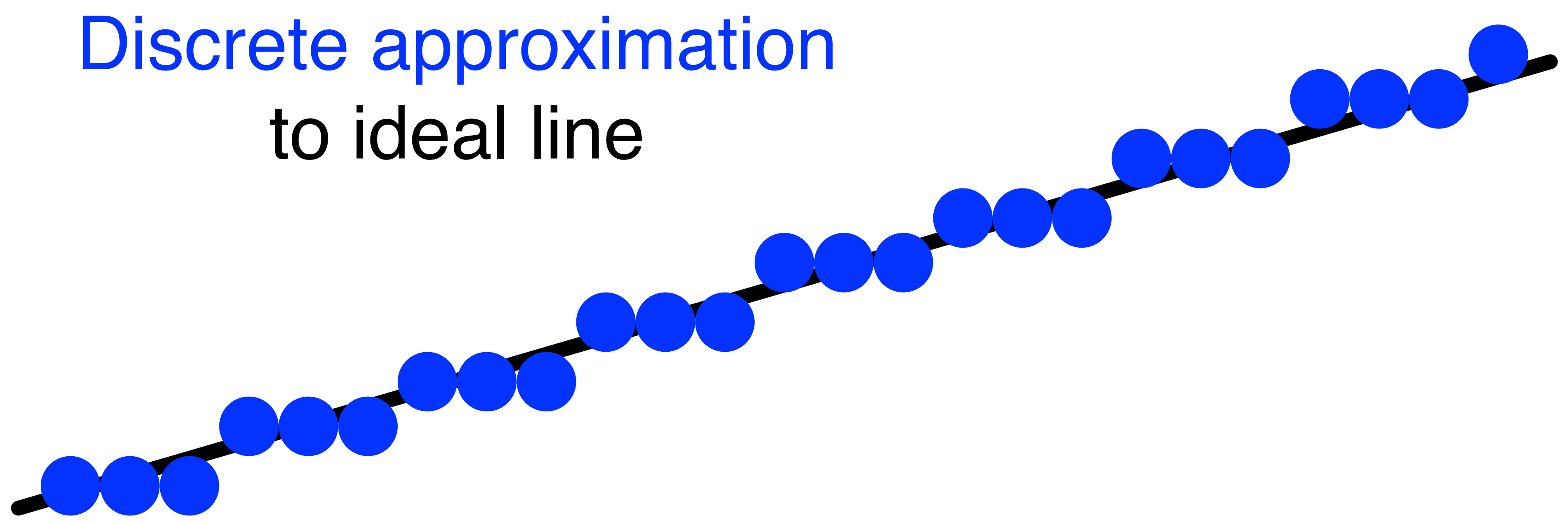
Rasterisation

- After projection, polygons are still described in continuous screen coordinates
- We need to use these polygons to colour in pixels on the screen



Rasterising lines

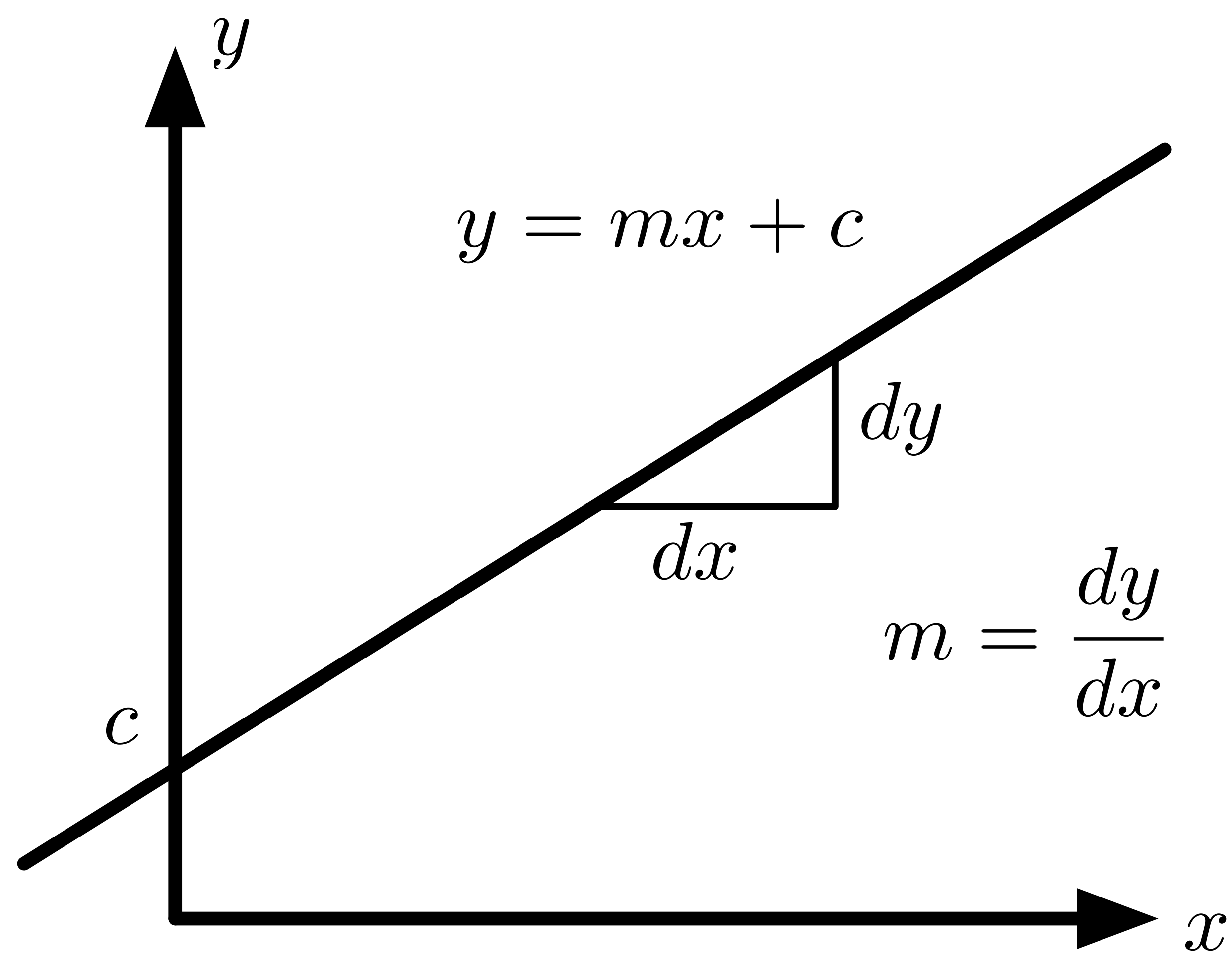
We need to convert a line described in a continuous coordinate system into a set of discrete pixels



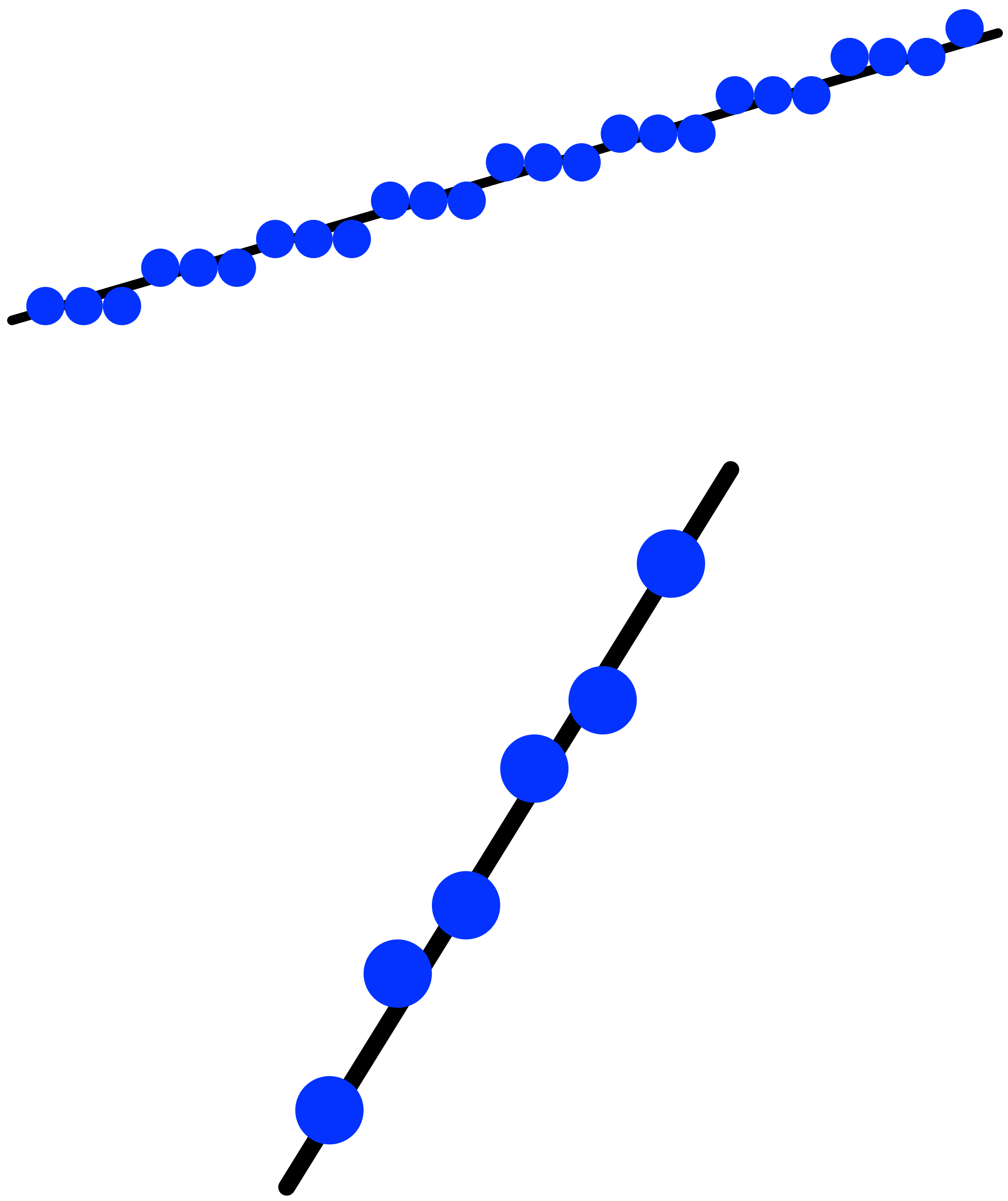
Simple line drawing

Linear algebra:
$y = mx + b$
A very simple approach:

- Increment x, calculate new y
- Cast x and y to integers



Simple line drawing

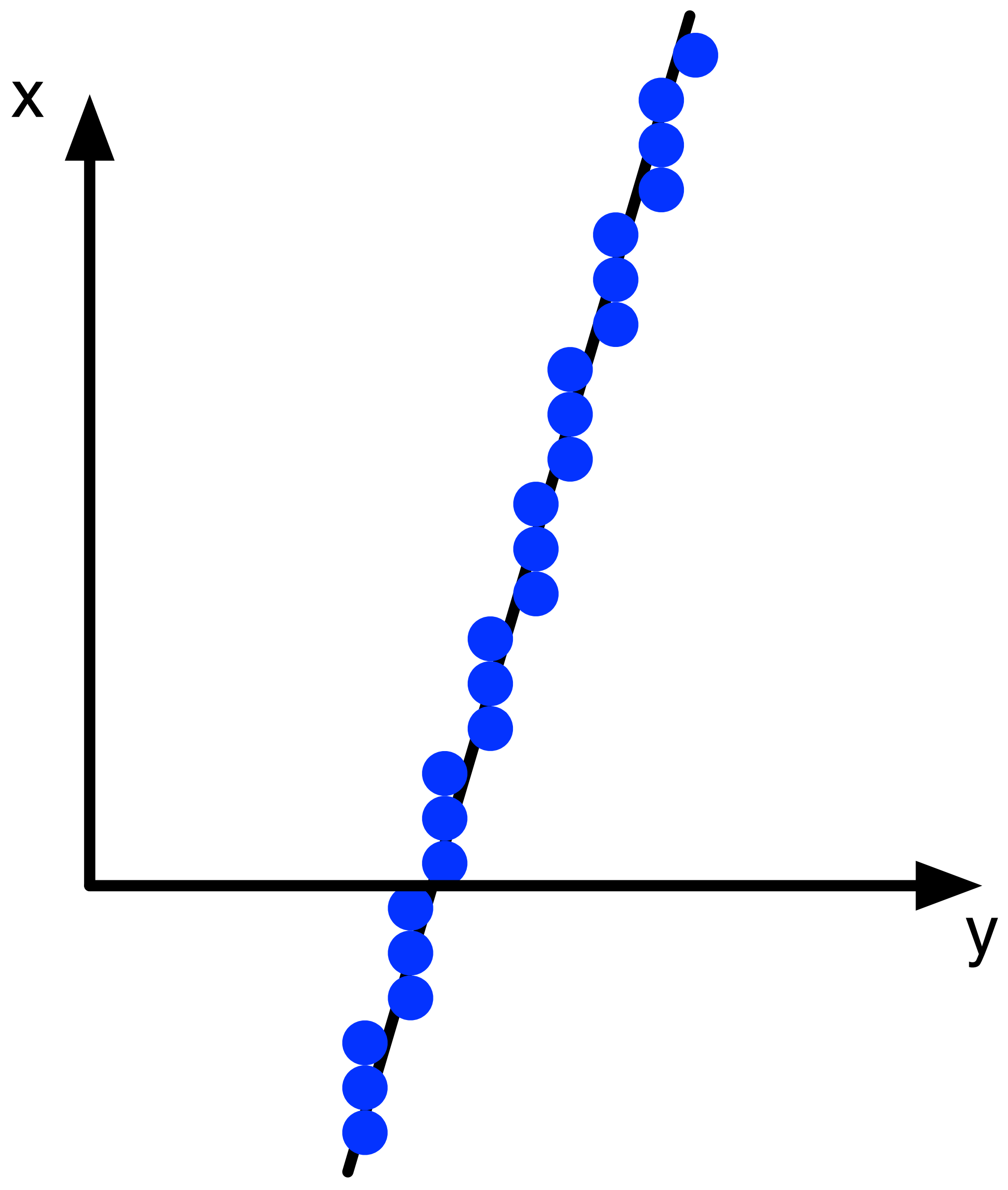


- For lines where $m \leq 1$ this seems to work well
- When $m > 1$ this doesn't work, the line becomes discontinuous

Symmetry

If $m \leq 1$ increment along the x axis, otherwise when $m > 1$, increment along y axis

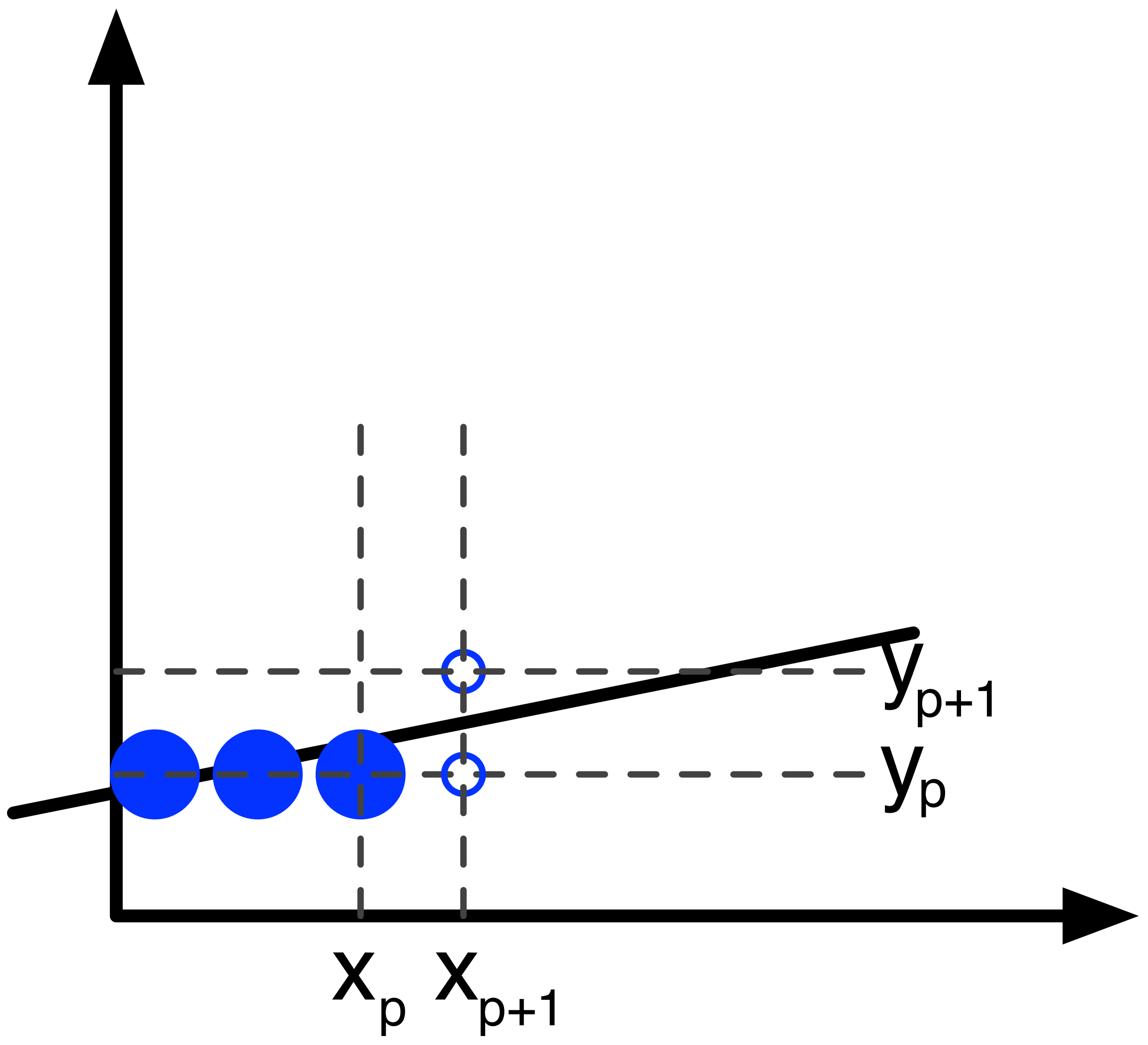
- This still requires a lot of floating point arithmetic



Midpoint algorithm

Iterate over steps, having lit a pixel (x_p, y_p) at step p:

- Check where the line intersects x_{p+1}
- Colour in (x_{p+1}, y_p) or (x_{p+1}, y_{p+1}) depending on which is closer to the intersection



Testing for the side of a line

- Assume the line is between (x_l, y_l) and (x_r, y_r)
- The slope of the line will be $\frac{dy}{dx}$ where $dx = x_r - x_l$ and $dy = y_r - y_l$

If $y = mx + c$ then $y = \frac{dy}{dx}x + c$ and so:

$$F(x, y) = ax + by + c = 0$$

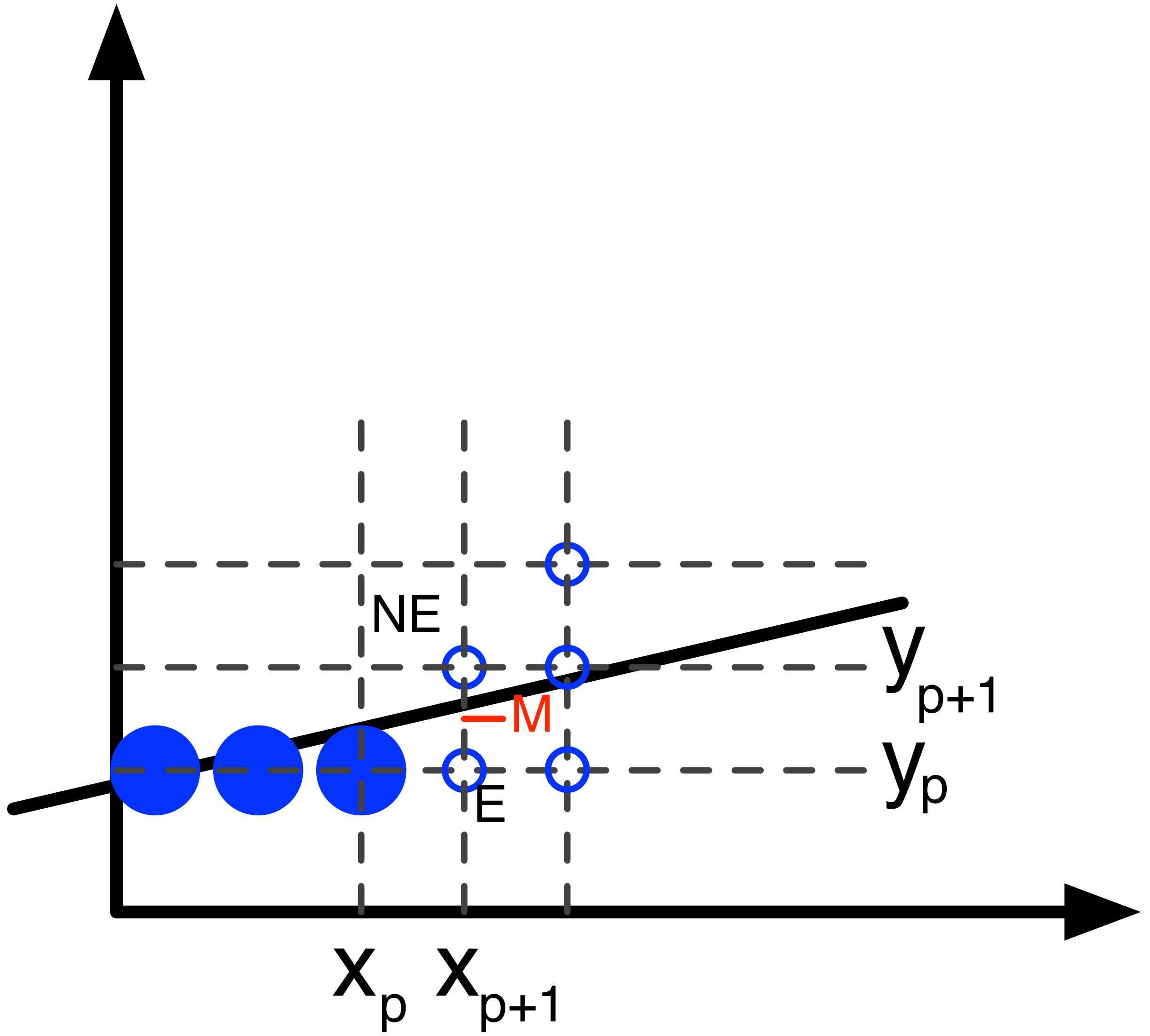
$$F(x, y) = dy.x - dx.y + c = 0$$

Decision variable

Assuming $\frac{dy}{dx} < 1$ using symmetry, evalulate F at point M:

$d = F(x_p + 1, y_p + \frac{1}{2}) = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

Then d is a *decision variable*, if $d \leq 0$ then E is chosen as the next pixel, otherwise NE is chosen.

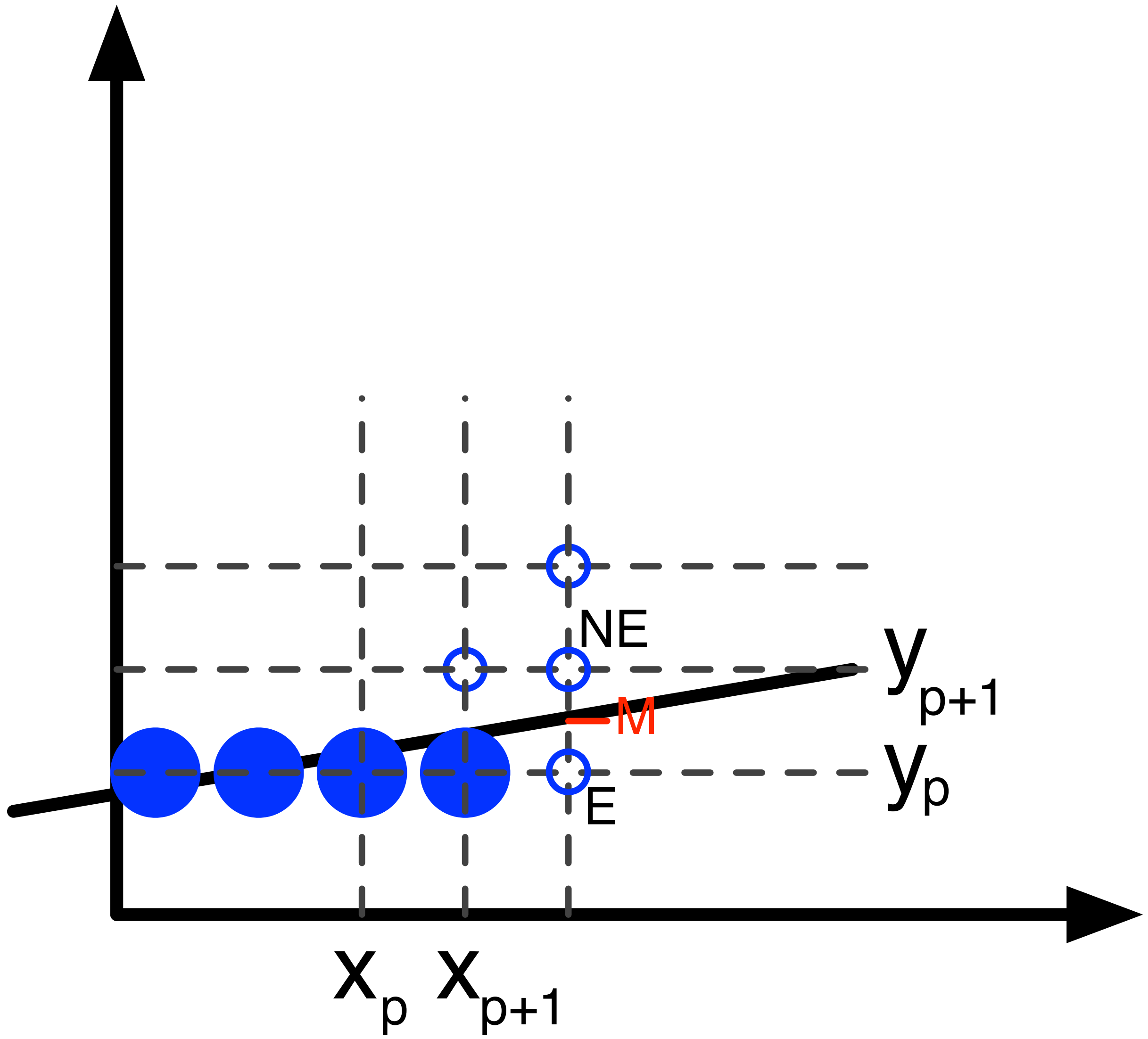


Updating the decision variable

Then to evaluate d for the next pixel, if we chose E:

$d' = F(x_p + 2, y_p + \frac{1}{2}) = a(x_p + 2) + b(y_p + \frac{1}{2}) + c$

Then since $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$, $d' = d + a = d + dy$

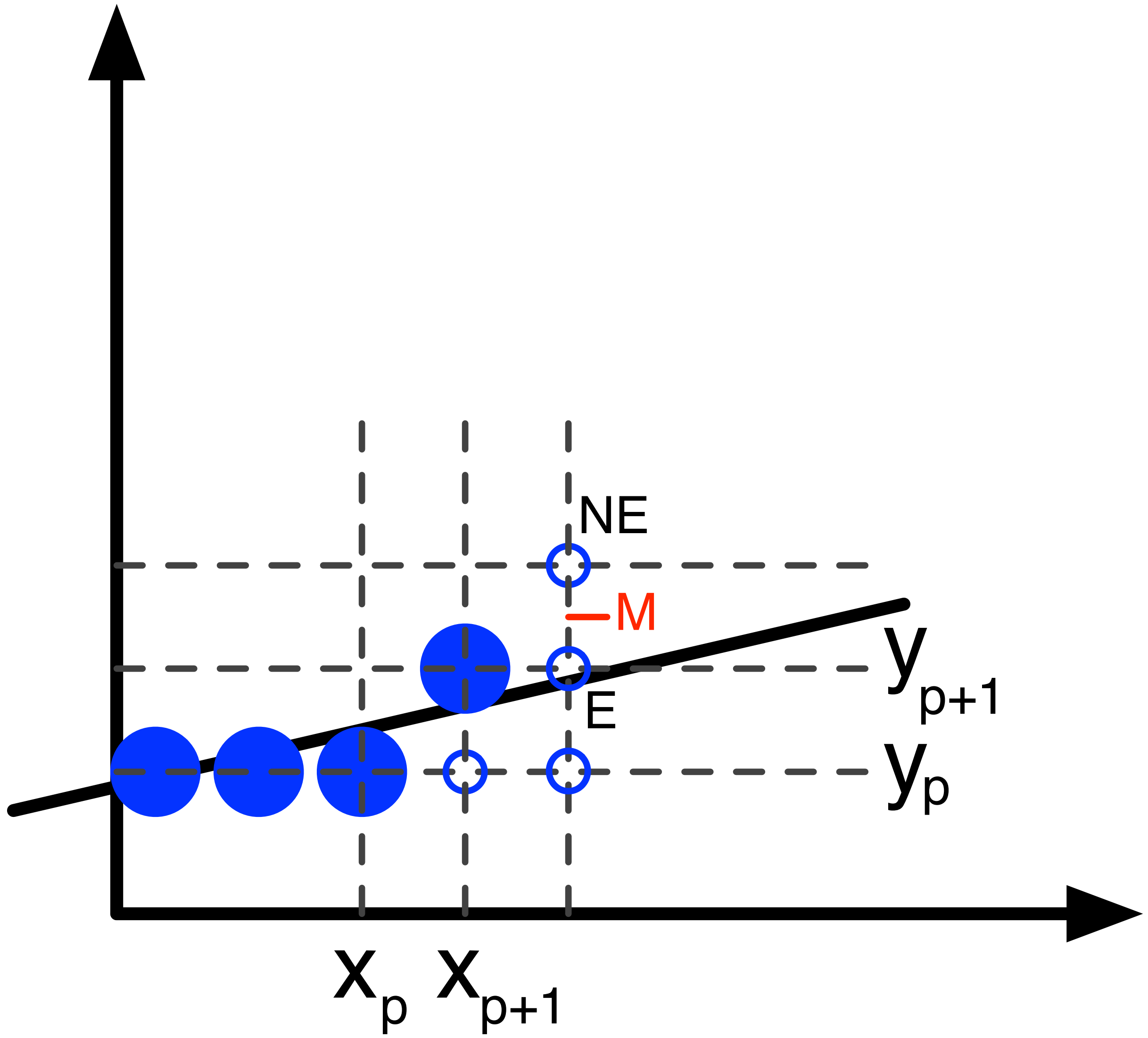


Updating the decision variable

If we chose NE:

$$d' = F(x_p + 2, y_p + \tfrac{3}{2}) = a(x_p + 2) + b(y_p + \tfrac{3}{2}) + c$$

Then since $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$,
$d' = d + a + b = d + dy - dx$



Initial value of d

The line starts from (x_l, y_l), so:

$$\begin{aligned} d_{start} &= F(x_l + 1, y_l + \frac{1}{2}) \\ &= a(x_l + 1) + b(y_l + \frac{1}{2}) + c \\ &= ax_l + by_l + c + a + \frac{b}{2} \\ &= F(x_l, y_l) + a + \frac{b}{2} \end{aligned}$$

But since (x_l, y_l) is on the line, $F(x_l, y_l) = 0$, so:

$d_{start} = dy - \frac{dx}{2}$

Then to avoid floating point operations, we multiply d by 2.

Decision variables

After we multiply by 2, $d = 2(ax + by + c)$.

$$\begin{aligned} d_{start} &= 2dy - dx \\ d'_E &= d + 2dy \\ d'_{NE} &= d + 2dy - 2dx \end{aligned}$$

Then we only need integer operations

Summary of the mid-point algorithm

- Start at first endpoint
- Calculate initial value for d
- Decide between two next pixels based on decision variable
- Update the decision based upon which pixel is chosen
- Iterate

Midpoint algorithm

```
void midpointLine(int x1, int y1, int x2, int y2)
{
int dx=x2-x1;
int dy=y2-y1;
int d=2*dy-dx;
int incrE=2*dy;
int incrNE=2*(dy-dx);
x=x1;
y=y1;
drawPixel(x,y);
```

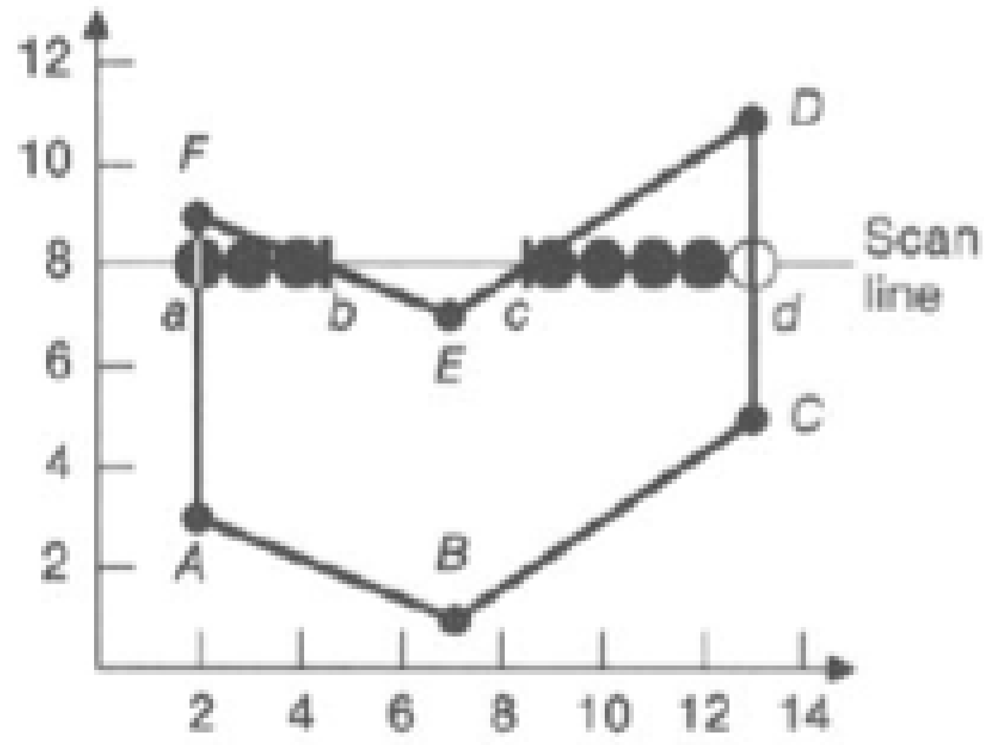
```
while{x < x2)
{
    if (d<=0) {
            d+=incrE;
            x++;
        }
    else
    {
        d+=incrNE;
        x++;
        y++;
    }
    drawPixel(x,y);
}
}
```

Overview

- Line rasterisation
- **Polygon rasterisation**
- Mean value coordinates
- Decomposing polygons

Scanline algorithm

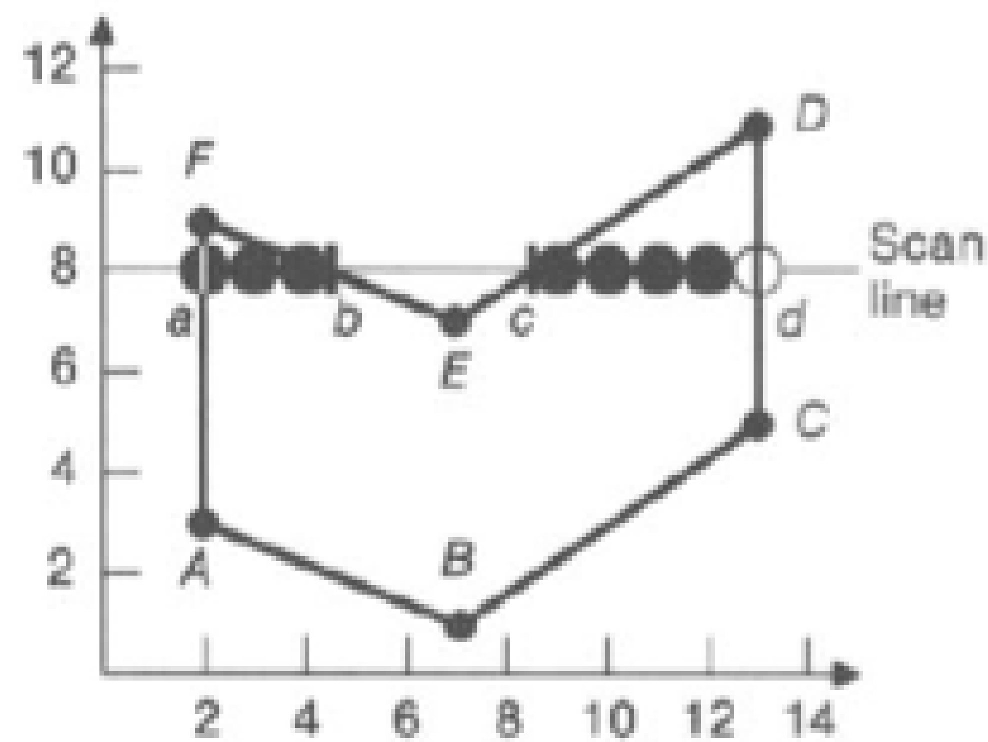
- Fill pixels within a polygon scanline by scanline



Scanline algorithm

On every scanline:

- Find intersections of scan line with all edges of the polygon
- Sort intersections in increasing order of x coordinate
- Fill in pixels between all pairs of intersections

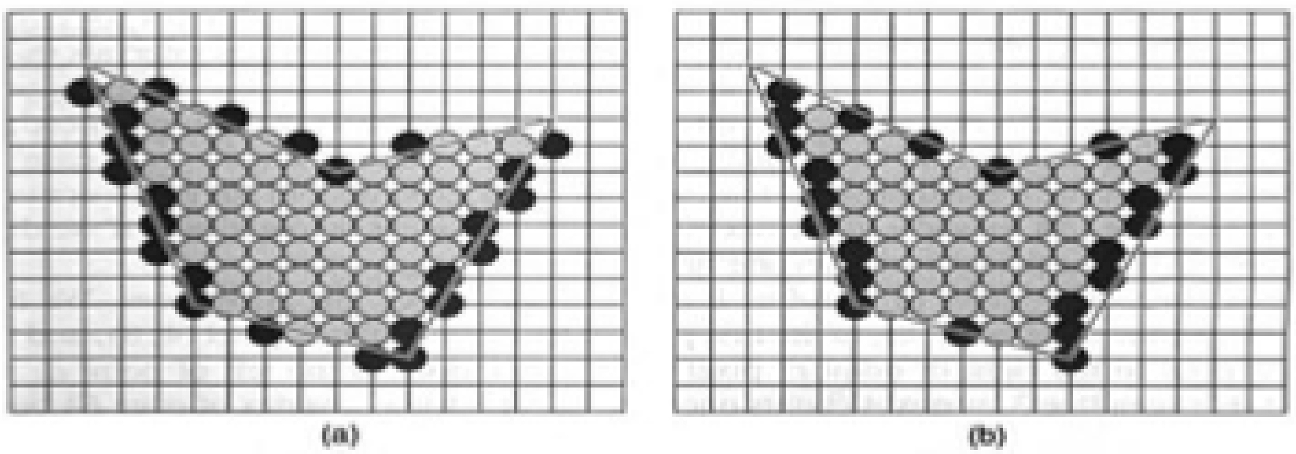


Works with concave polygons

Span extrema

Only turn on pixels that have their centre interior to the polygon

- Otherwise pixels overlap with adjacent polygons



This is done by rounding up values on left edges and down on right edges

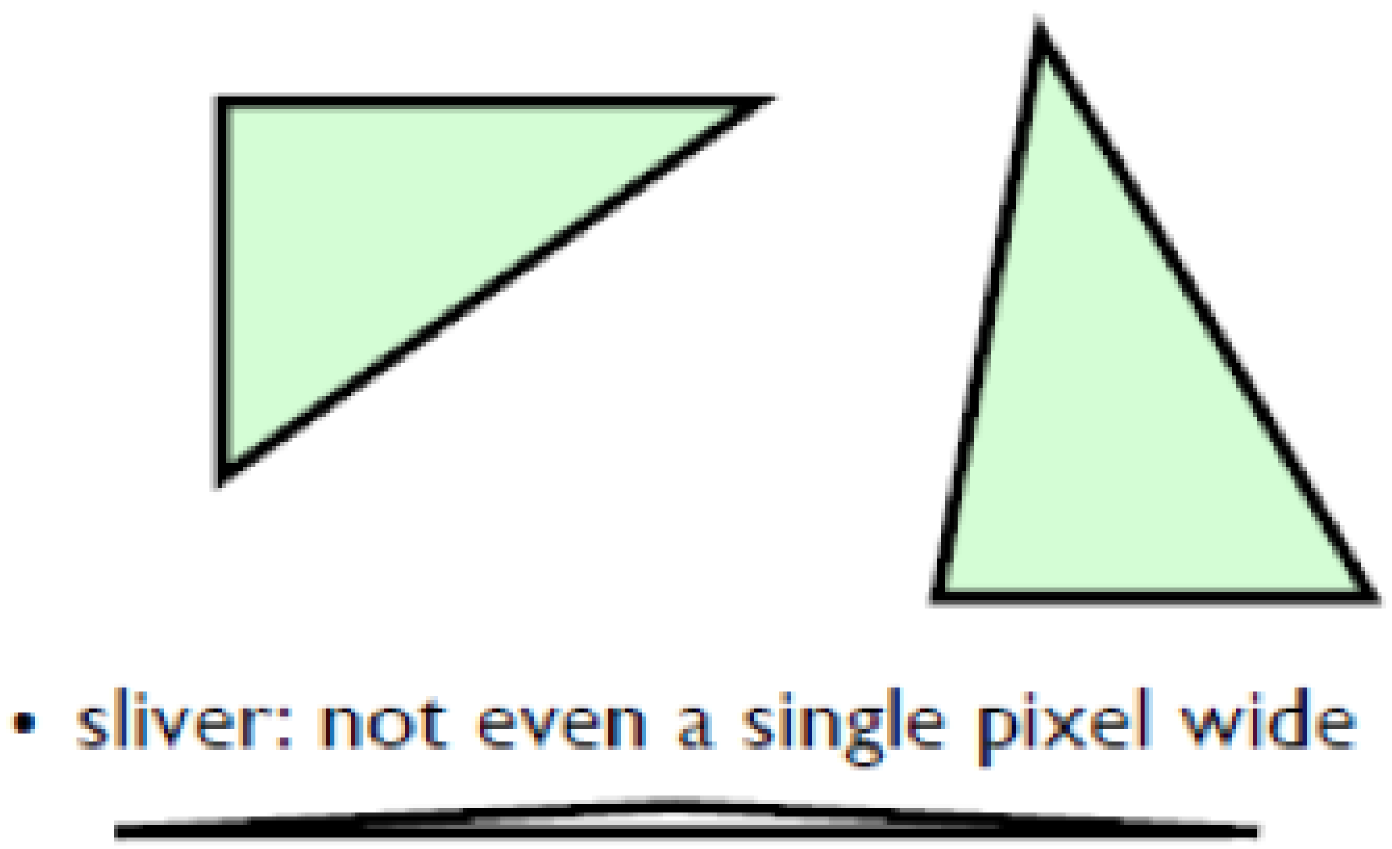
Scanline algorithm

Pros:

- Simple

Cons:

- Hard to parallelise efficiently
- Special cases can occur and require exception handling



Barycentric coordinates for triangles

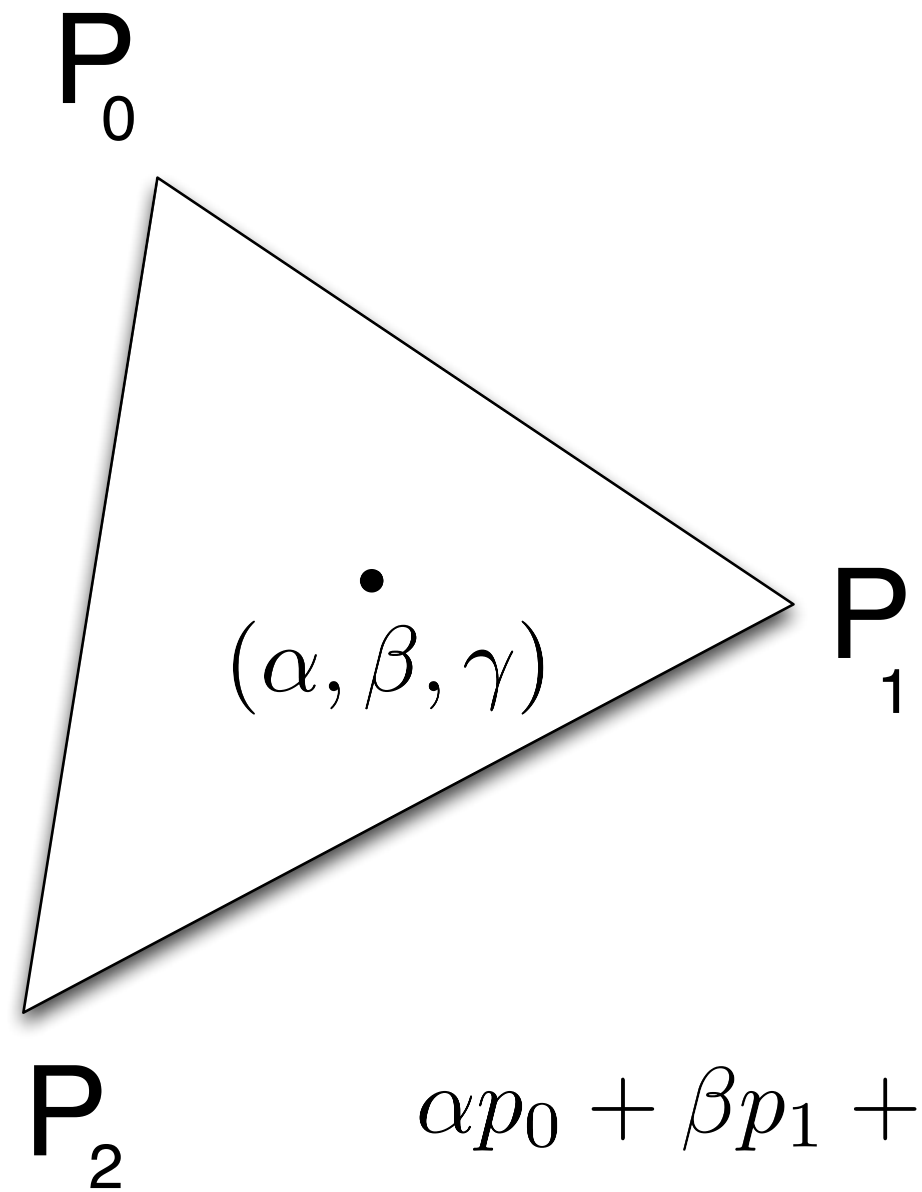
- Allow us to check whether a pixel is inside or outside a triangle
- Makes it easy to interpolate attributes between vertices
- Used in GPUs
- Easy to parallelise

Barycentric coordinates for triangles

Given a 2D triangle with vertices p_0, p_1, p_2. For any point in the plane p:

$$
\begin{aligned}
p &= p_0 + \beta(p_1 - p_0) + \gamma(p_2 - p_0) \\
&= (1 - \beta - \gamma)p_0 + \beta p_1 + \gamma p_2 \\
&= \alpha p_0 + \beta p_1 + \gamma p_2
\end{aligned}
$$

$$\alpha + \beta + \gamma = 1$$

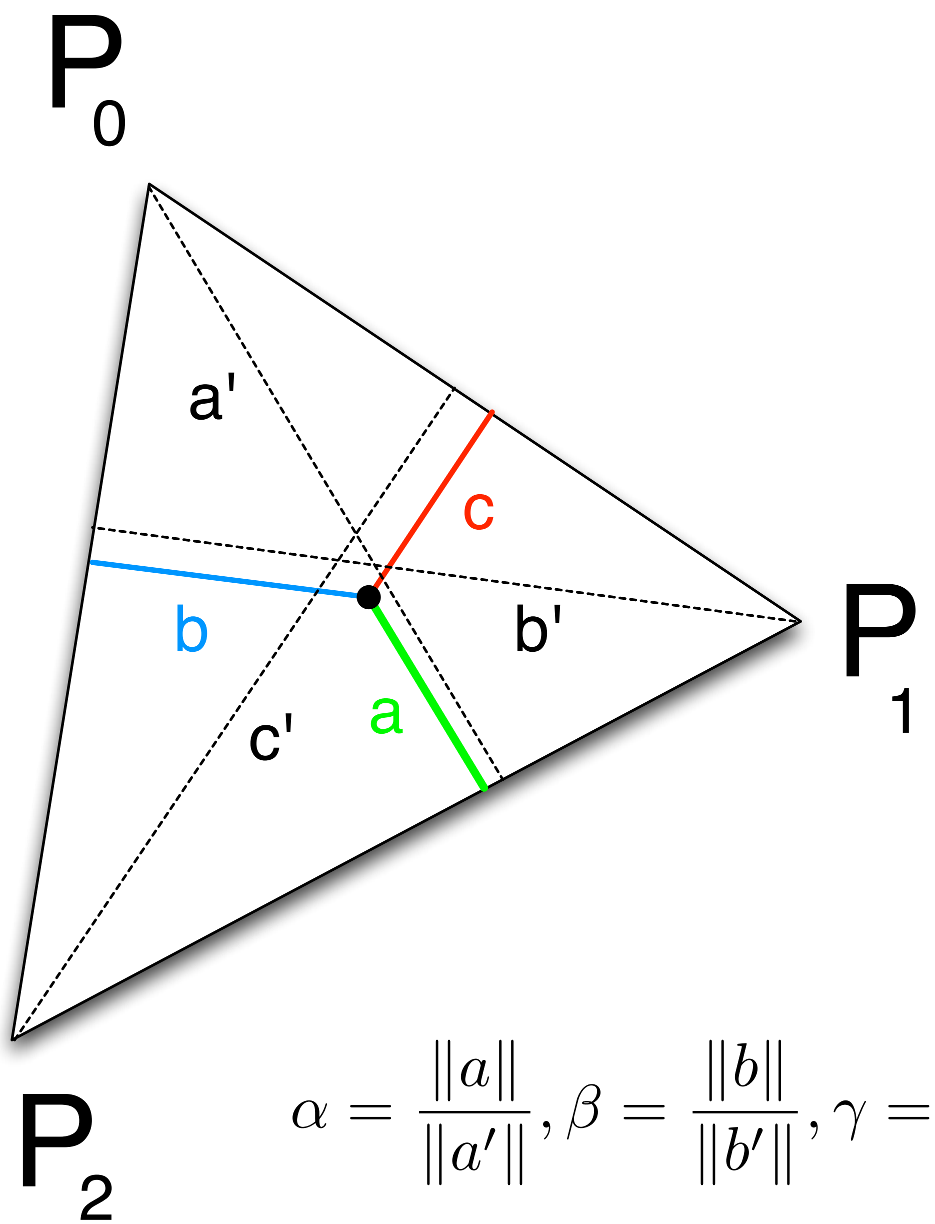


$$\alpha p_0 + \beta p_1 + \gamma p_2$$

Barycentric coordinates for triangles

The values $\alpha, \beta, \gamma \in [0, 1]$ if and only if *p* is inside the triangle.

α, β, γ are the *barycentric coordinates* of the point *p*.



$$\alpha = \frac{\|a\|}{\|a'\|}, \beta = \frac{\|b\|}{\|b'\|}, \gamma = \frac{\|c\|}{\|c'\|}$$

Calculating barycentric coordinates

If the triangle is composed of $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, then for a point (x, y):

$\alpha = \frac{f_{12}(x,y)}{f_{12}(x_0,y_0)}$, $\beta = \frac{f_{20}(x,y)}{f_{20}(x_1,y_1)}$ $\gamma = \frac{f_{01}(x,y)}{f_{01}(x_2,y_2)}$

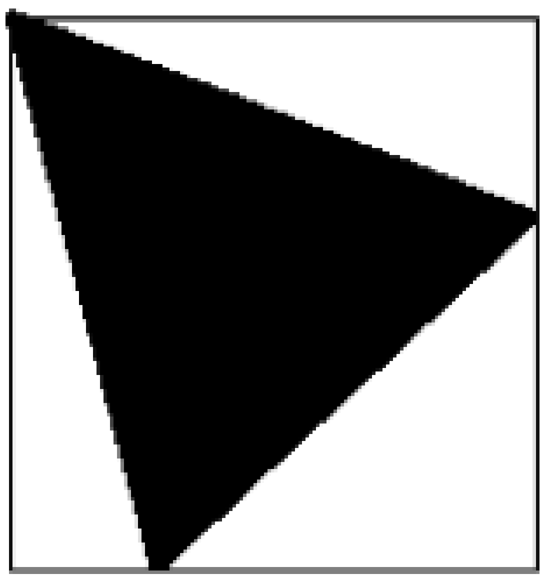
where $f_{ab} = (y_a - y_b)x + (x_b - x_a)y + x_a y_b - x_b y_a$

Bounding box of a triangle

We calculate a bounding box around the triangle, by taking the minimum and maximum vertex coordinates in each direction:

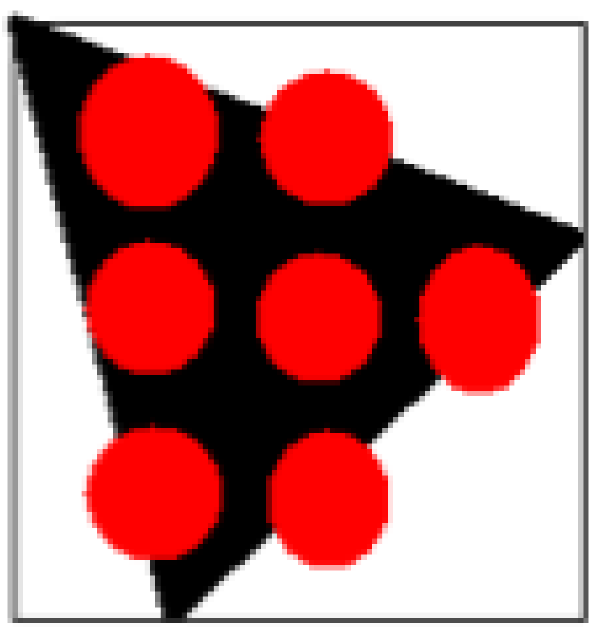
$$x_{min}, y_{min} = \min(x_0, x_1, x_2), \min(y_0, y_1, y_2)$$

$$x_{max}, y_{max} = \max(x_0, x_1, x_2), \max(y_0, y_1, y_2)$$



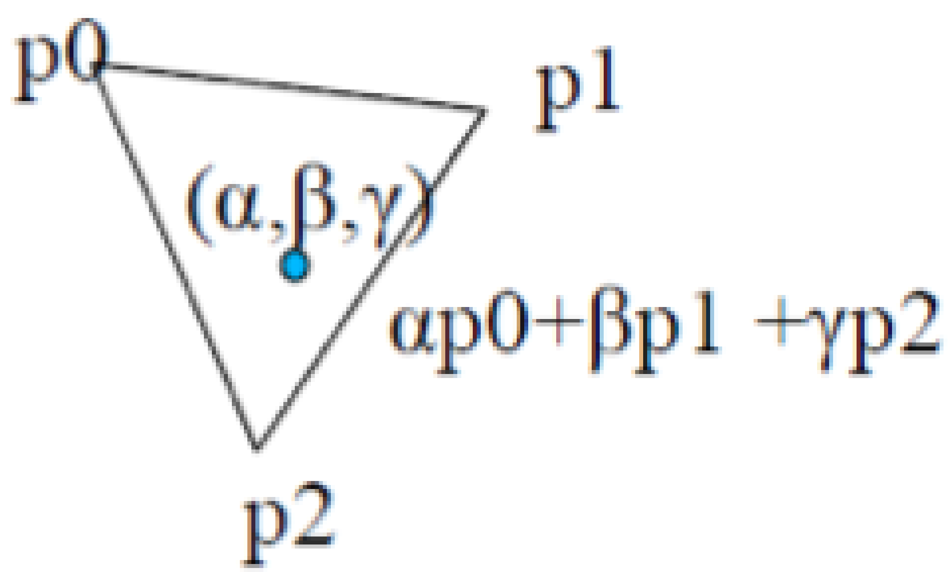
Scanning inside the triangle

- For each pixel in the bounding box, compute the barycentric coordinates
- Shade the pixel if all three values $\alpha, \beta, \gamma \in [0, 1]$



Interpolation

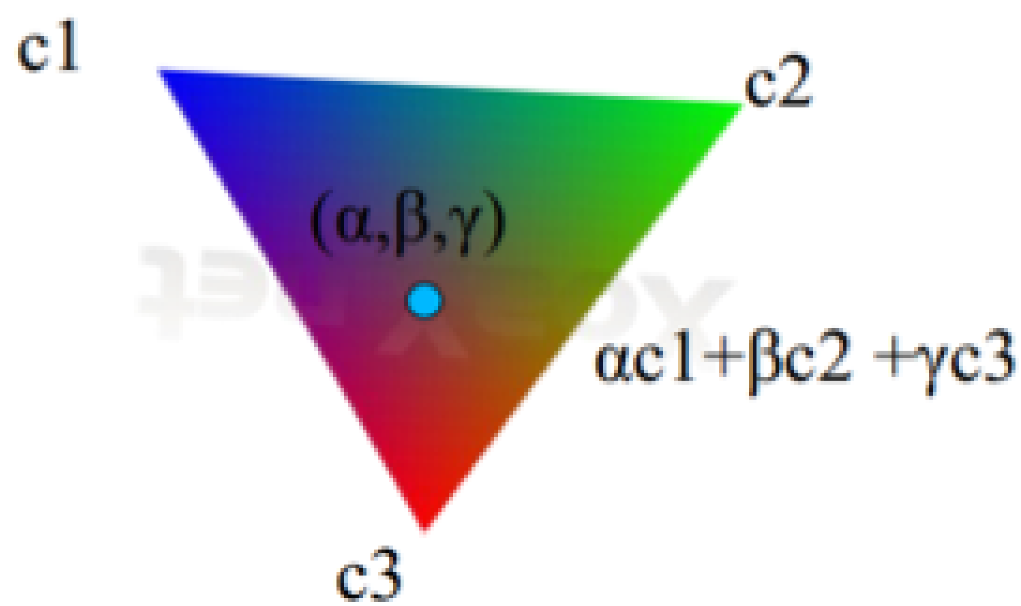
Barycentric coordinates can be used to interpolate attributes of triangle vertices, for example colour, depth, normal vectors or texture coordinates.



Interpolation of colour

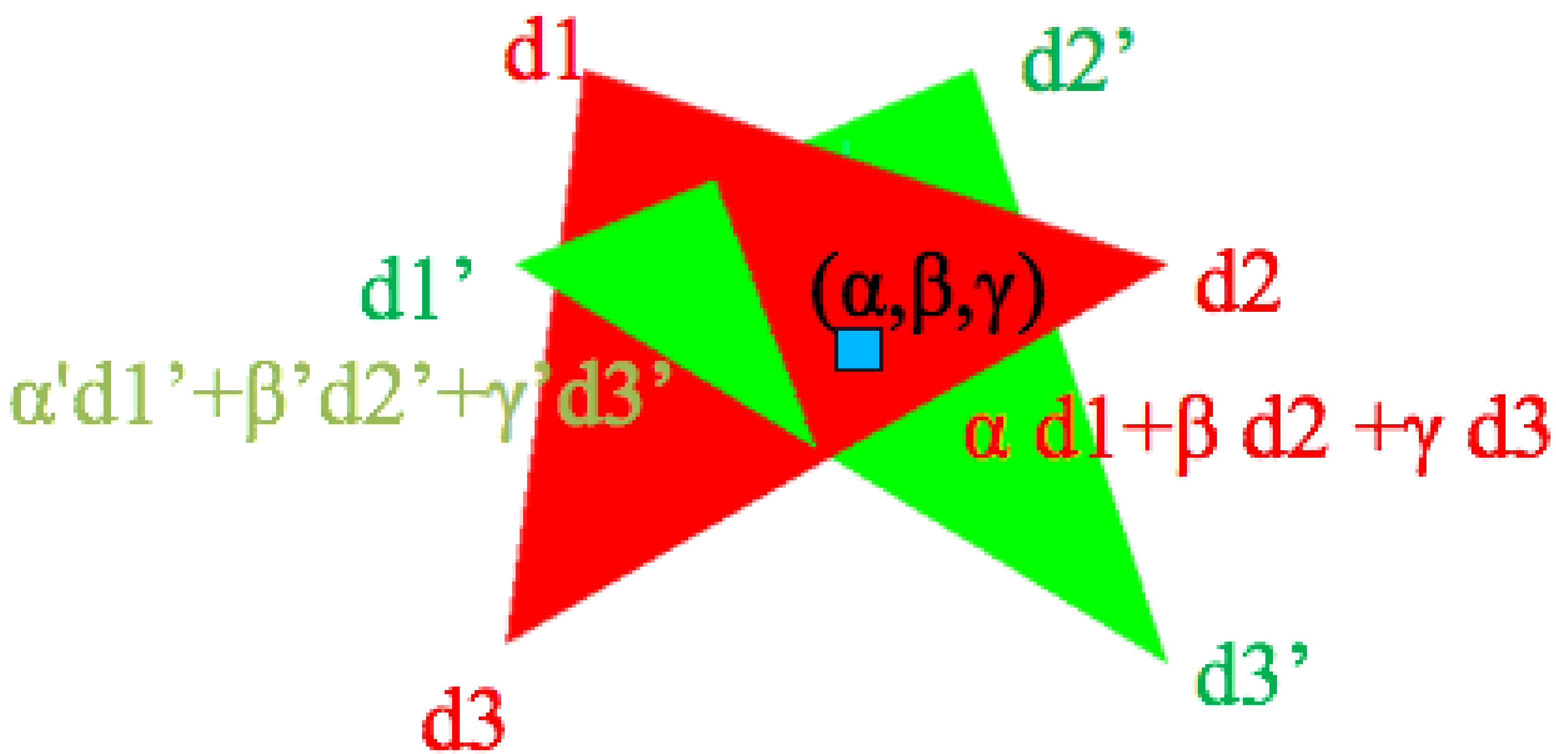
Gouraud shading:

- Calculate colour at vertices and interpolate the colour over the surface

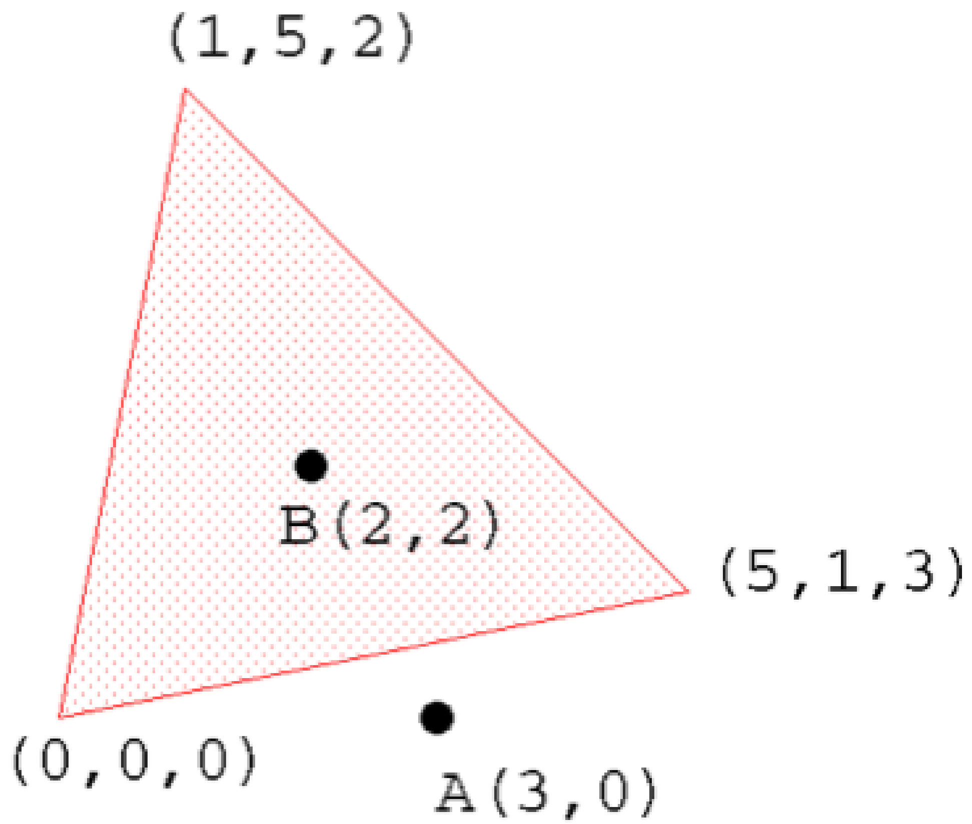


Interpolation of depth

- When triangles overlap each other, depth needs to be calculated at each pixel in case the intersect
- Calculate using barycentric coordinates
- Used in Z-buffering



Exercise

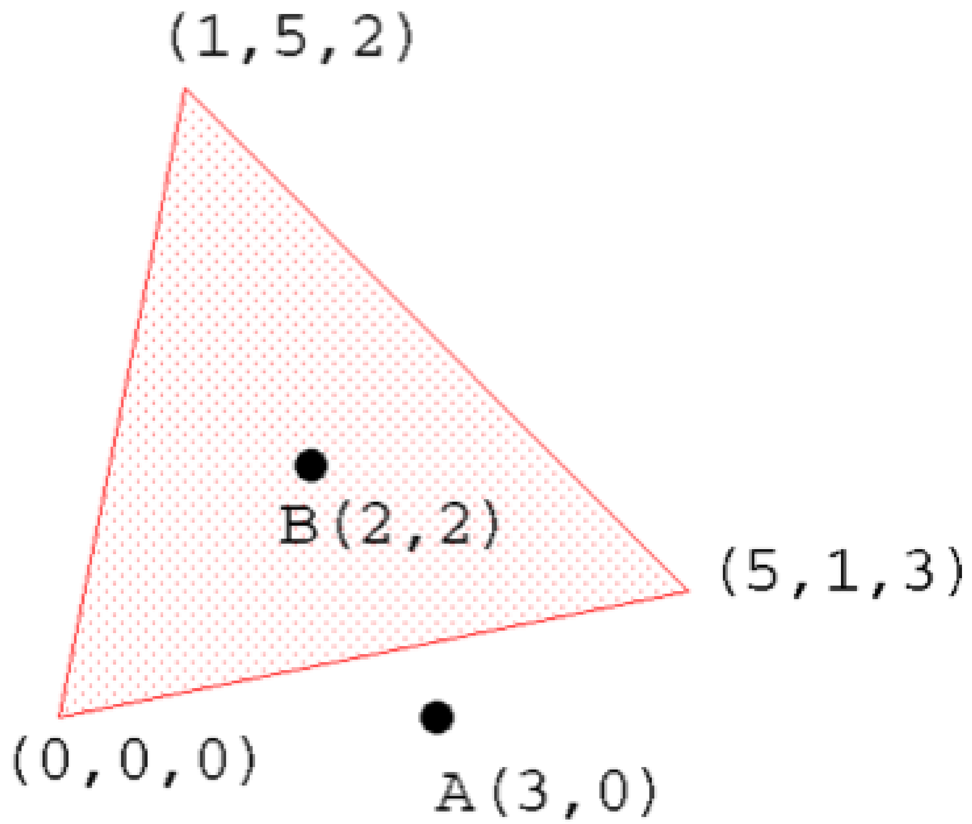


- What are the barycentric coordinates of A and B?
- What is the surface depth (Z coordinate) at B

$$\gamma = \frac{(y_0 - y_1)x + (x_1 - x_0)y + x_0 y_1 - x_1 y_0}{(y_0 - y_1)x_2 + (x_1 - x_0)y_2 + x_0 y_1 - x_1 y_0}$$

$$\beta = \frac{(y_0 - y_2)x + (x_2 - x_0)y + x_0 y_2 - x_2 y_0}{(y_0 - y_2)x_2 + (x_2 - x_0)y_2 + x_0 y_2 - x_2 y_0}$$

Exercise



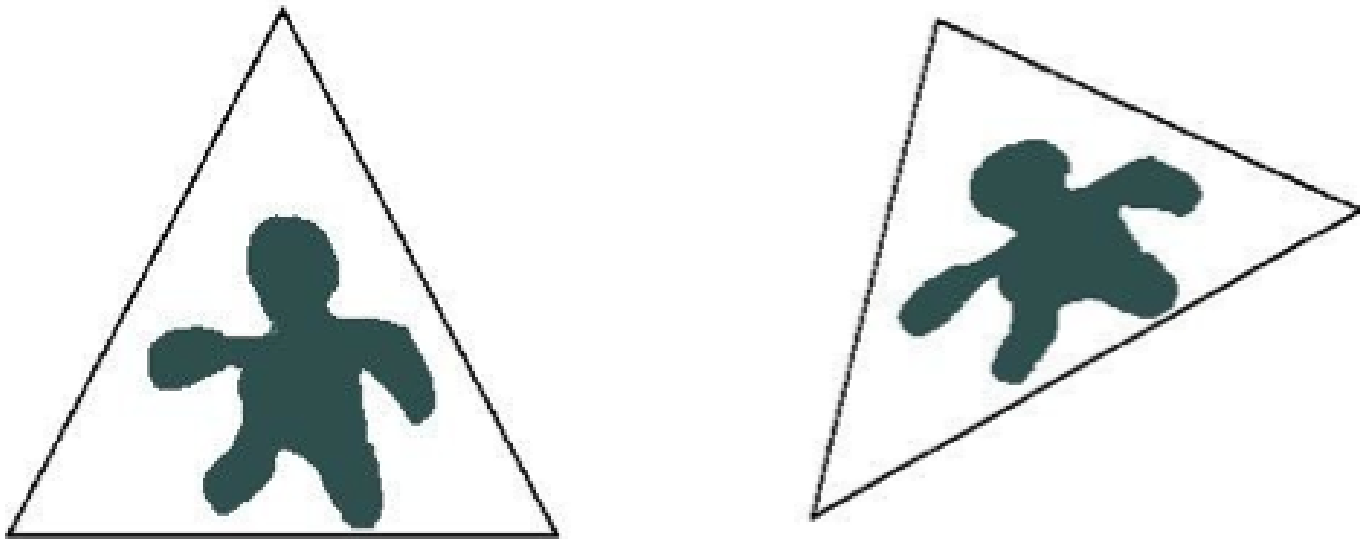
Barycentric coordinates

- $A = (\frac{1}{2}, \frac{5}{8}, -\frac{1}{8})$
- $B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Depth at $B = \frac{5}{3}$

Shape editing

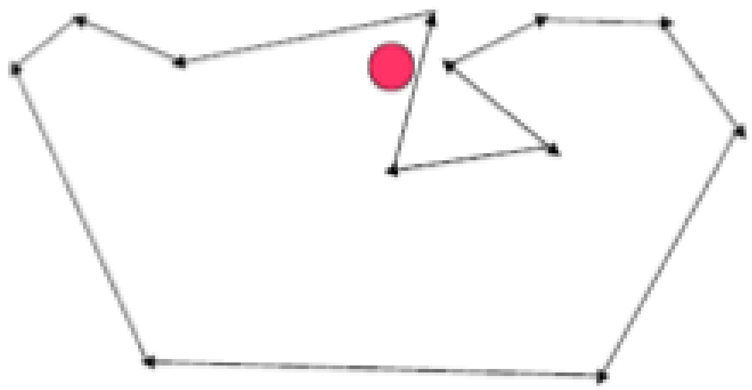
We can apply the same barycentric coordinates within a triangle when its shape is edited



General polygons

Barycentric coordinates for polygons with more vertices:

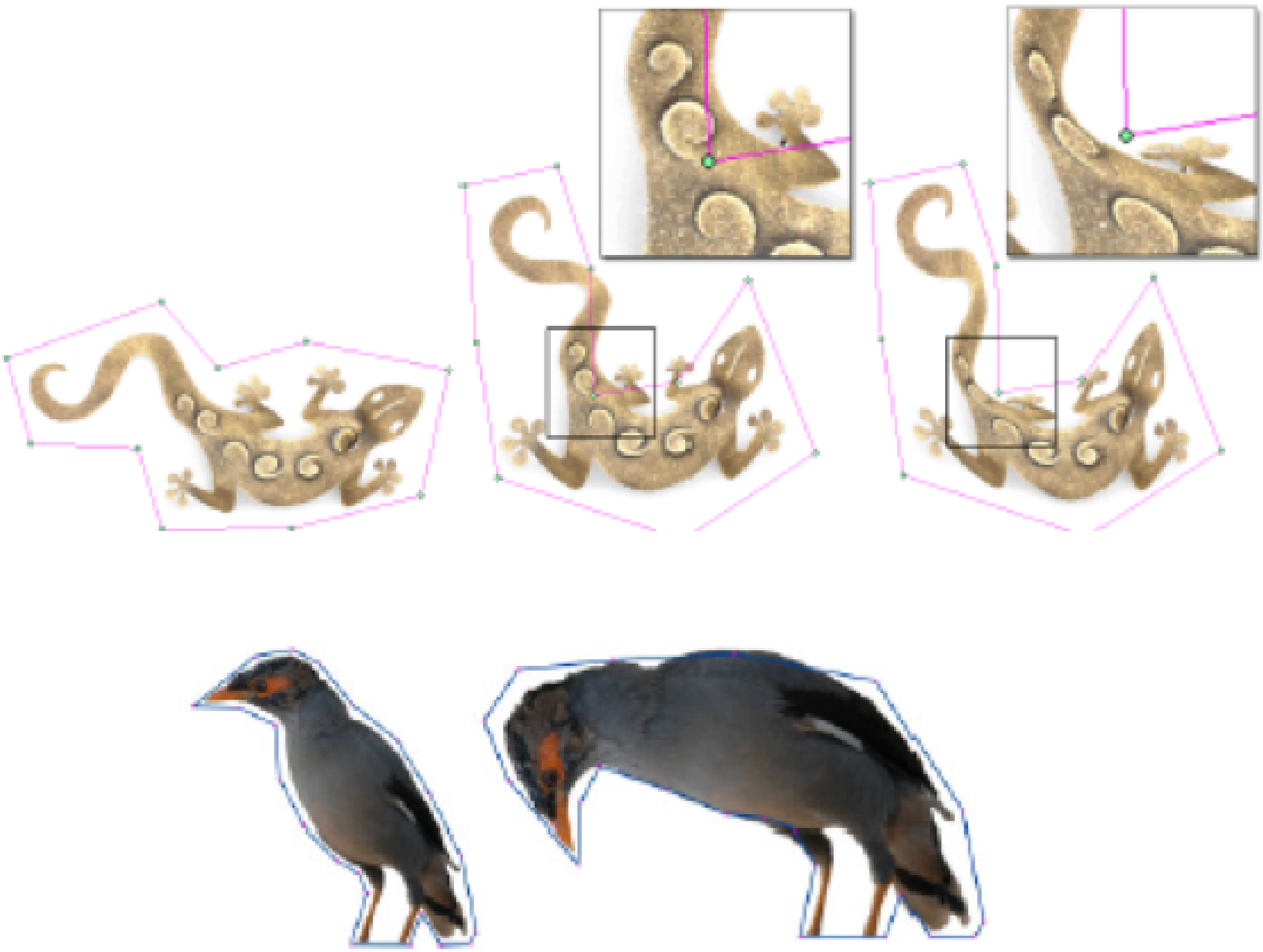
$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$



Barycentric coordinates for 3D meshes:

- Mean value coordinates
- Harmonic coordinates (generalised barycentric coordinates)

Shape editing

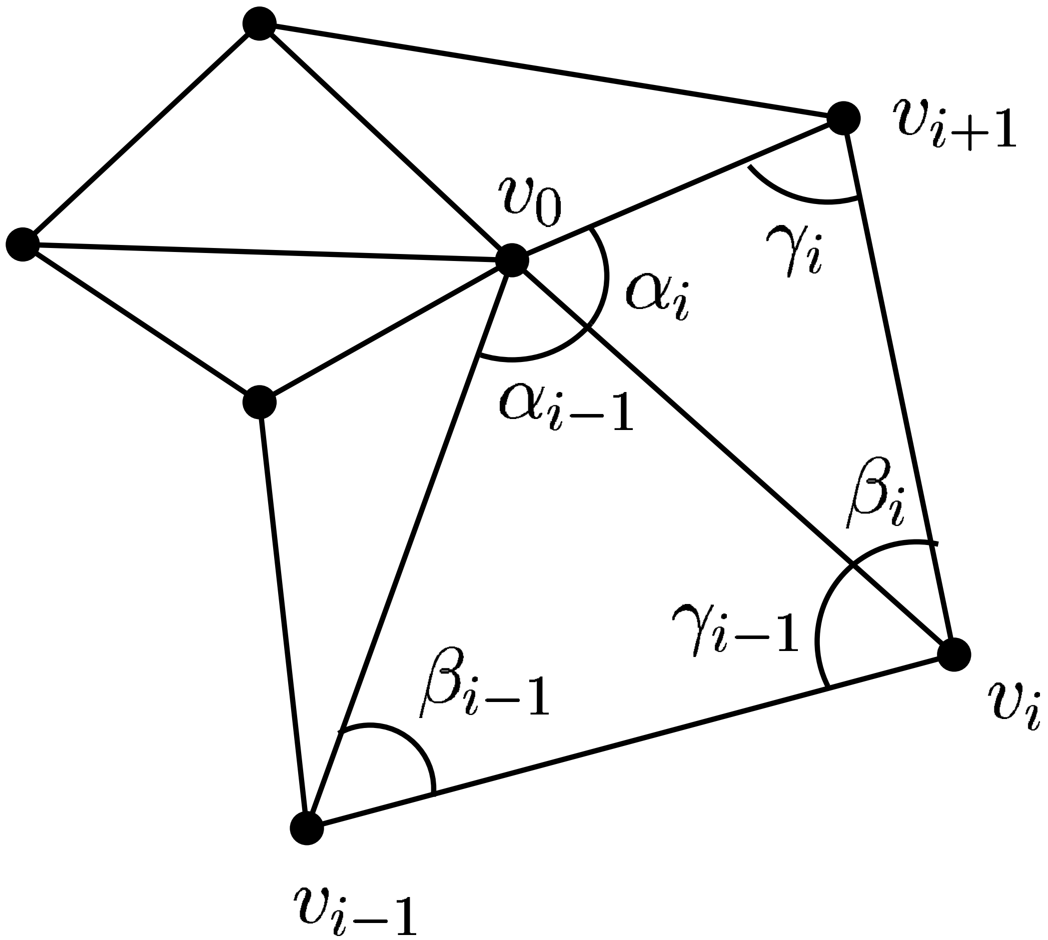


Mean value coordinates

Coordinates that can:

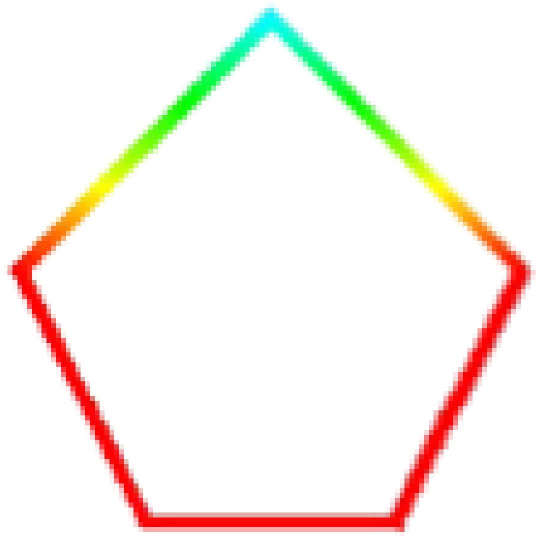
- smoothly interpolate boundary values
- works with concave polygons
- works in 3D

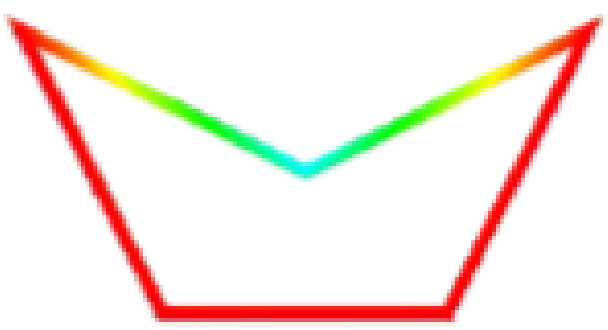
$$w_i = \frac{\tan \alpha_{i-1}/2 + \tan \alpha_i/2}{\|v_i - v_0\|}$$



Mean value coordinates

- Can interpolate convex and concave polygons
- Smoothly interpolates the interior as well as the exterior





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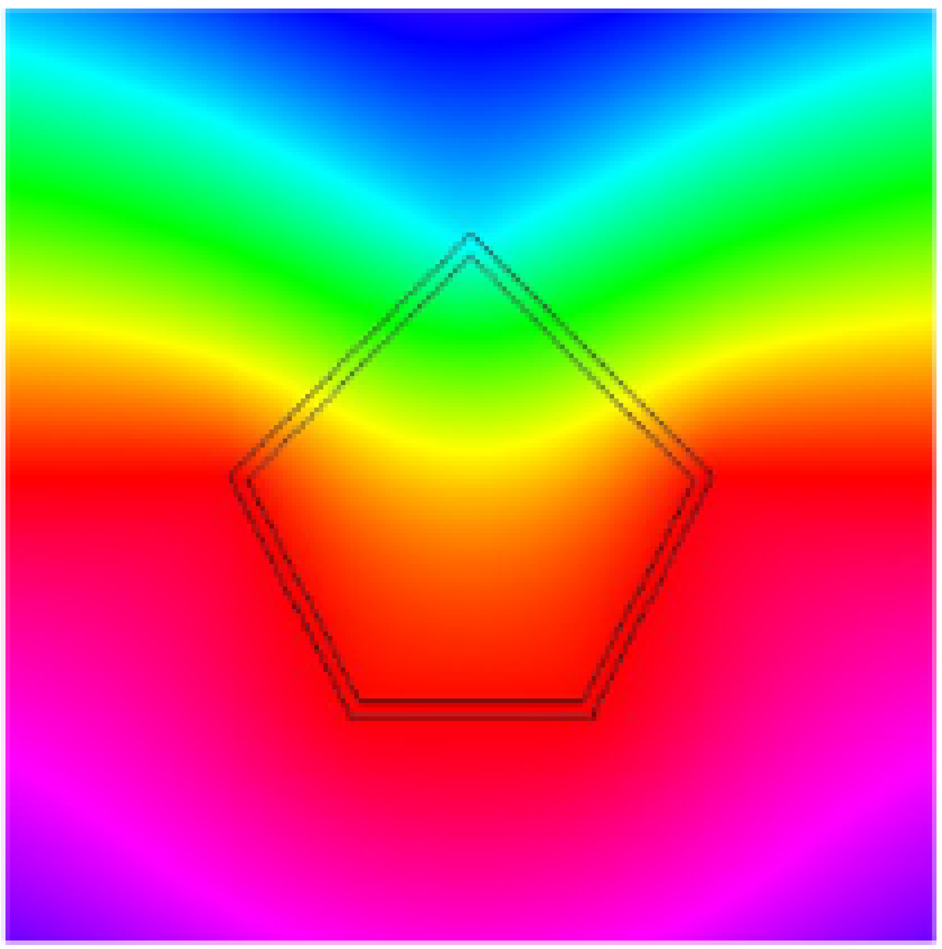
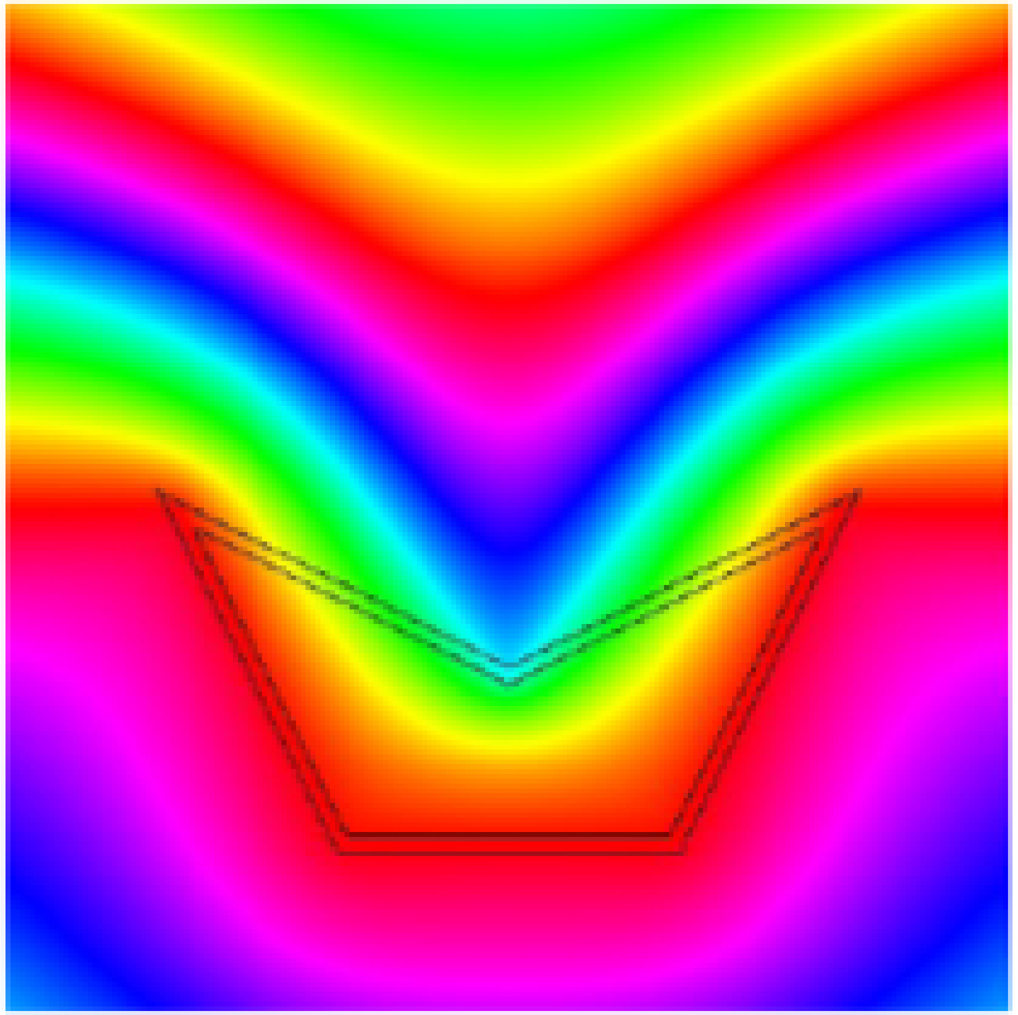
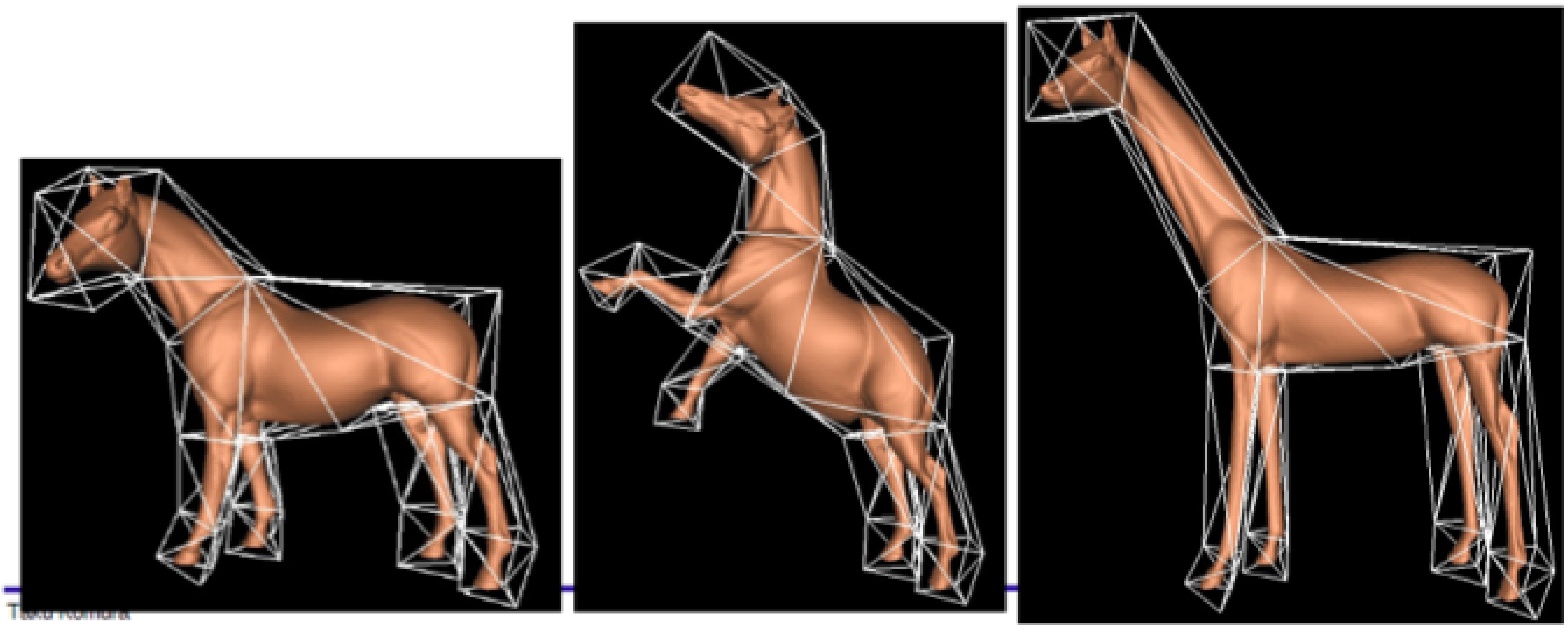




Mean value coordinates

- Can be computed in 3D
- Applicable for mesh editing



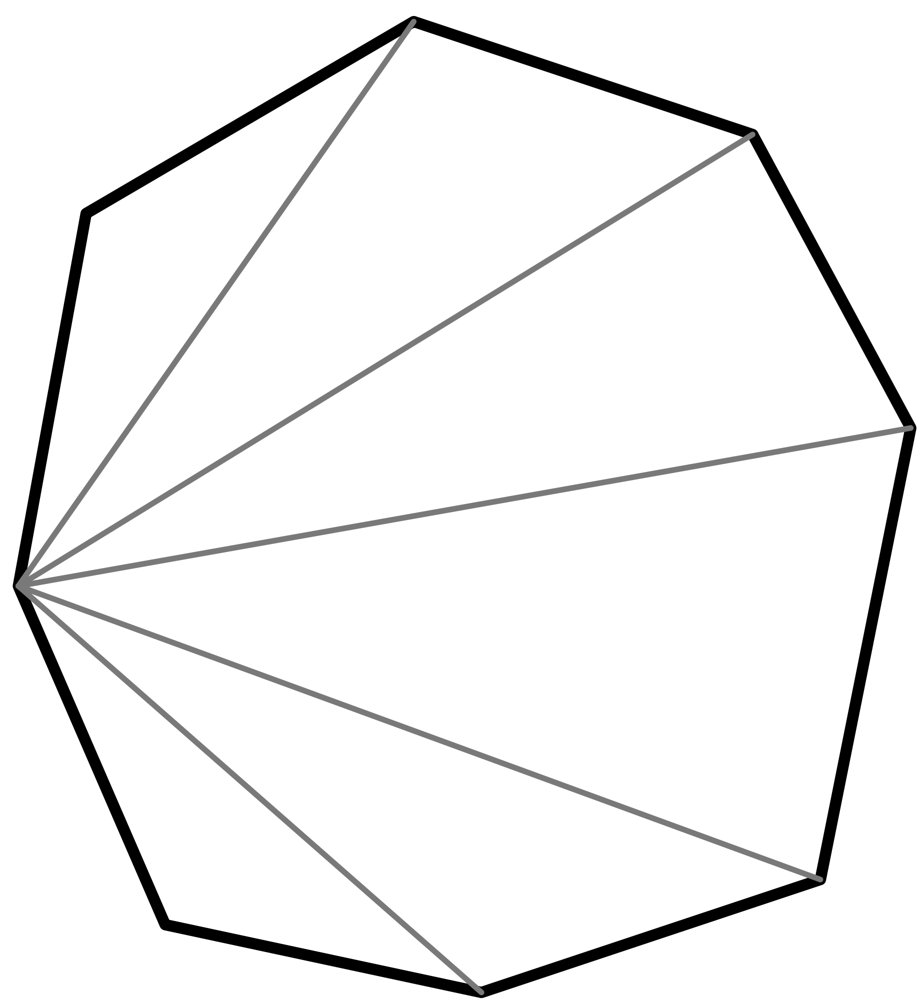
Overview

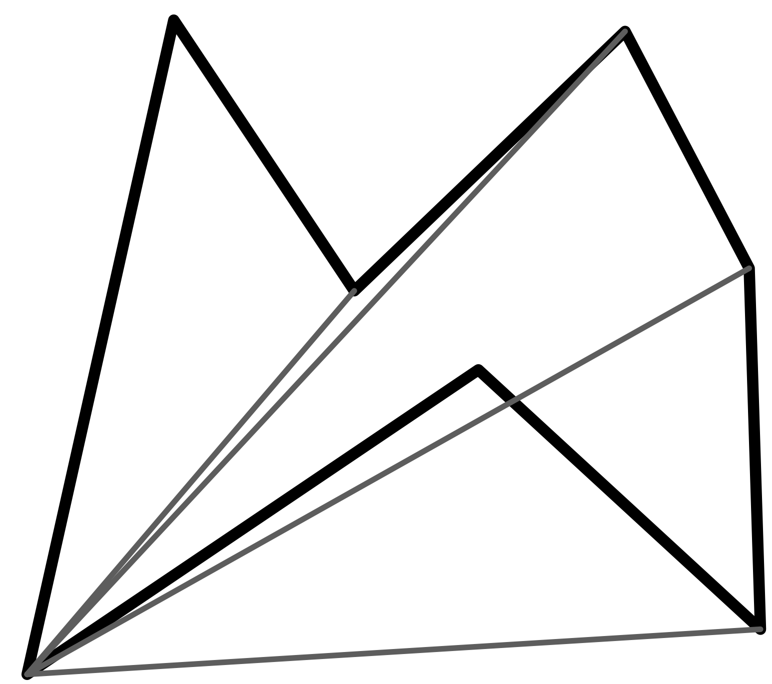
- Line rasterisation
- Polygon rasterisation
- Mean value coordinates
- **Decomposing polygons**

Polygon decomposition

For polygons with more than three vertices, we usually decompose them into triangles

Simple for convex





Concave is difficult

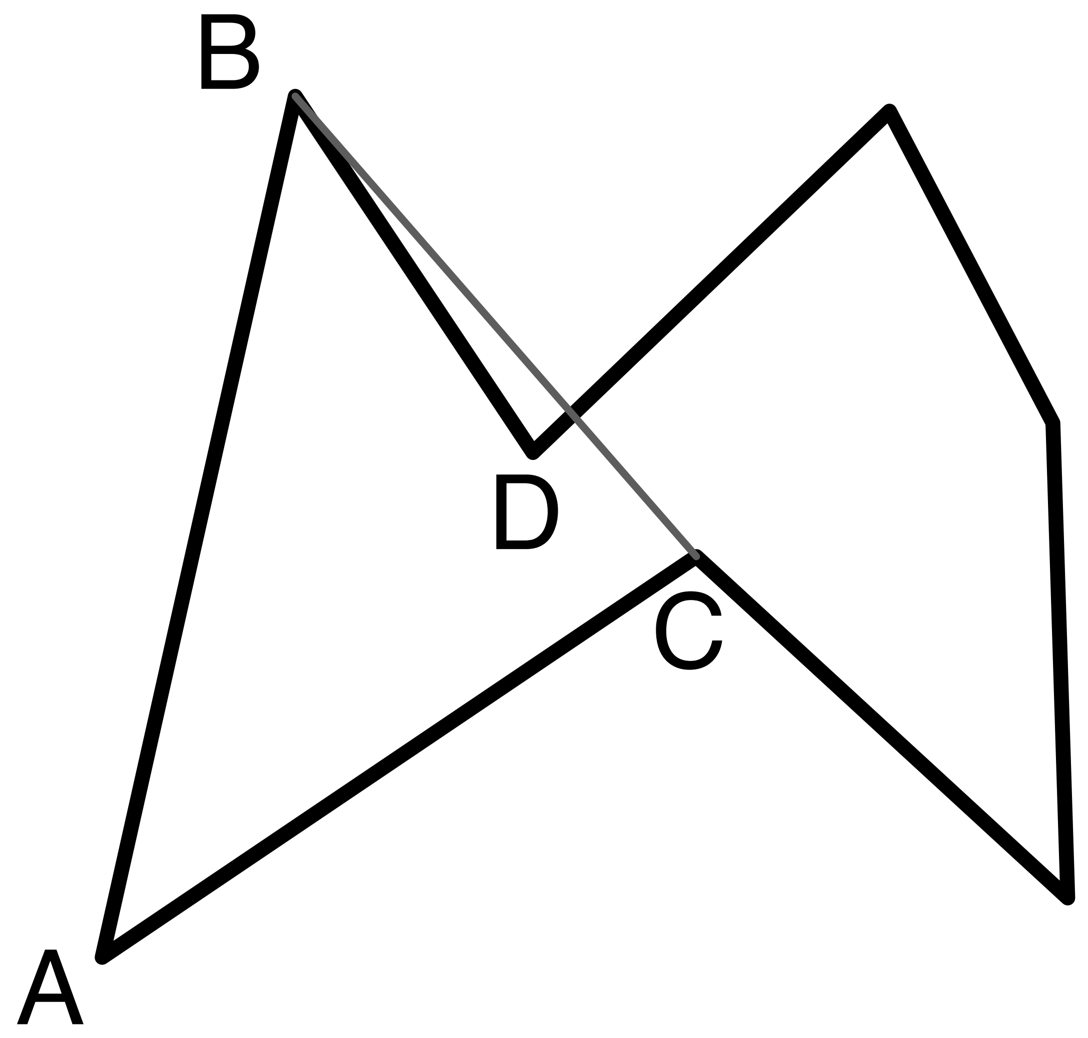
Polygon decomposition

Algorithm:

- Find leftmost vertex and label it A
- Compose potential triangle out of A and adjacent vertices B and C
- Check to see if another point of the polygon is inside the triangle ABC
- If all other points are outside ABC, remove ABC from the polygon and proceed with next leftmost vertex

Polygon decomposition

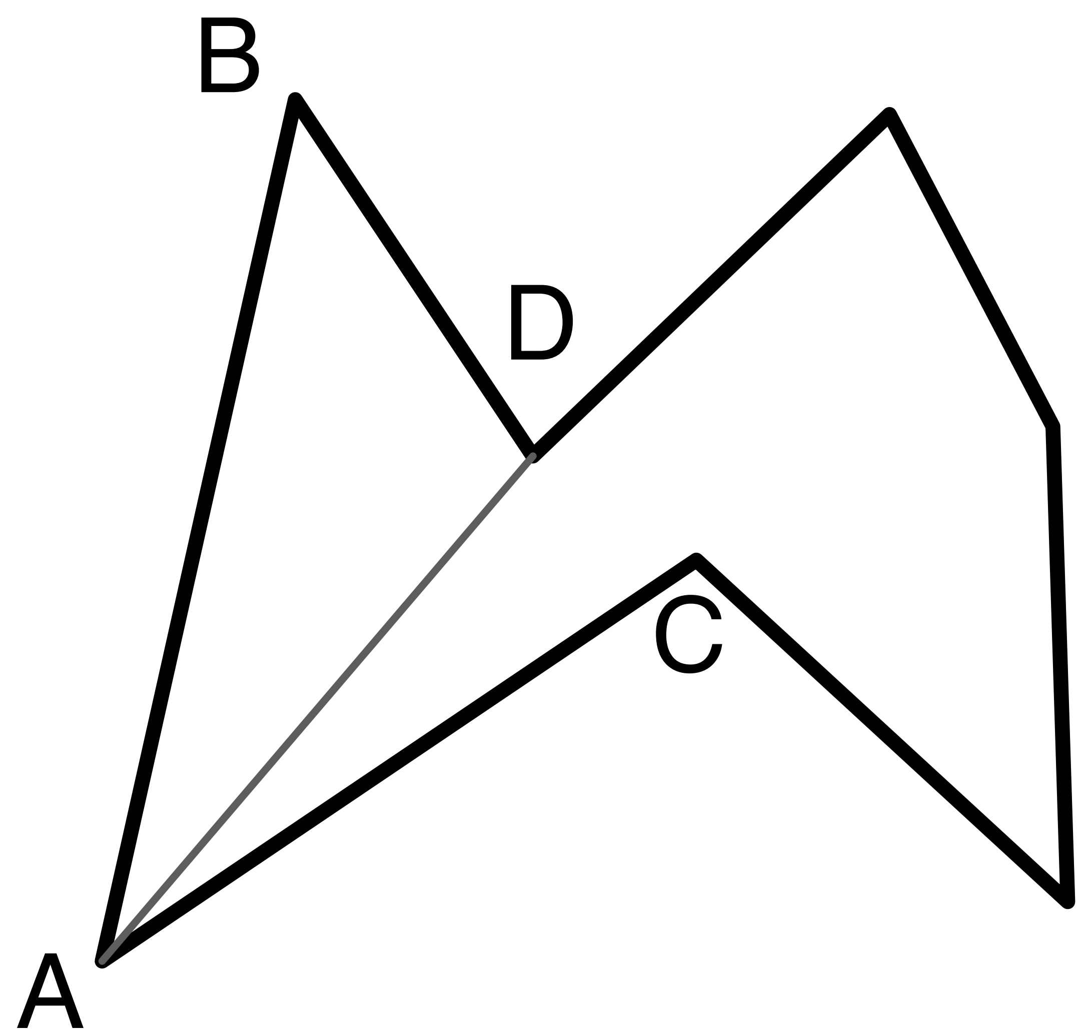
- Left most vertex is A
- Form a triangle between A and the adjacent B and C
- Check if all other vertices are outside this triangle



Vertex D fails test

Polygon decomposition

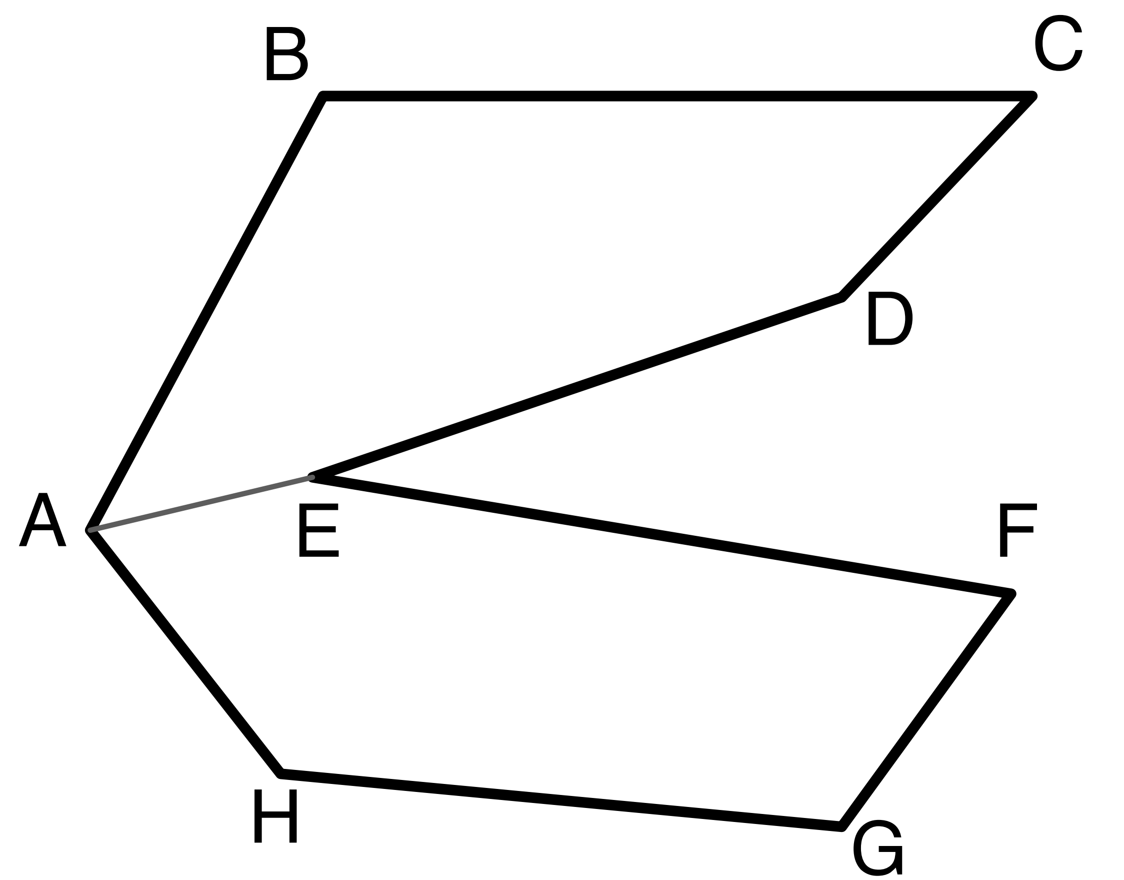
If a vertex is inside, split the polygon by the inside vertex and point A and continue:



Test ABD as before

Polygon decomposition

The new edge may split the polygon in two. If so recurse over each polygon:



Split into ABCDE and AEFGH

Summary

Rasterisation:

- Line rasterisation, midpoint algorithm
- Triangle rasterisation, scanline algorithm, barycentric coordinates
- Mean value coordinates
- Polygon decomposistion into triangles

References

Midpoint and scanline algorithm:

- Foley Chapter 3.2, 3.5, 3.6

Barycentric coordinates:

- Shirley Chapter 2.7

Mean value coordinates:

- Floater, M. S. Mean value coordinates. Computer Aided Geometric Design, 20(1), 19–27, 2003
- Ju, T., Schaefer, S., & Warren, J. Mean value coordinates for closed triangular meshes. ACM Transactions on Graphics, 24(3), 561–566, 2005.

Polygon decomposition:

- `http://www.siggraph.org/education/materials/HyperGraph/scanline/outprims/polygon1.htm`