Computer Graphics 3 - Transformations

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Slides courtesy of Taku Komura
www.inf.ed.ac.uk/teaching/courses/cg

Overview

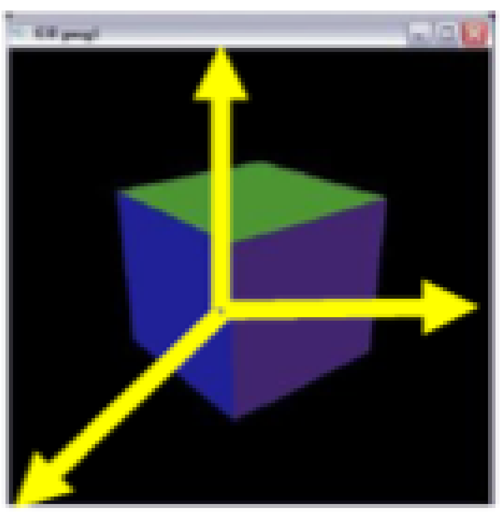
Homogeneous transformations

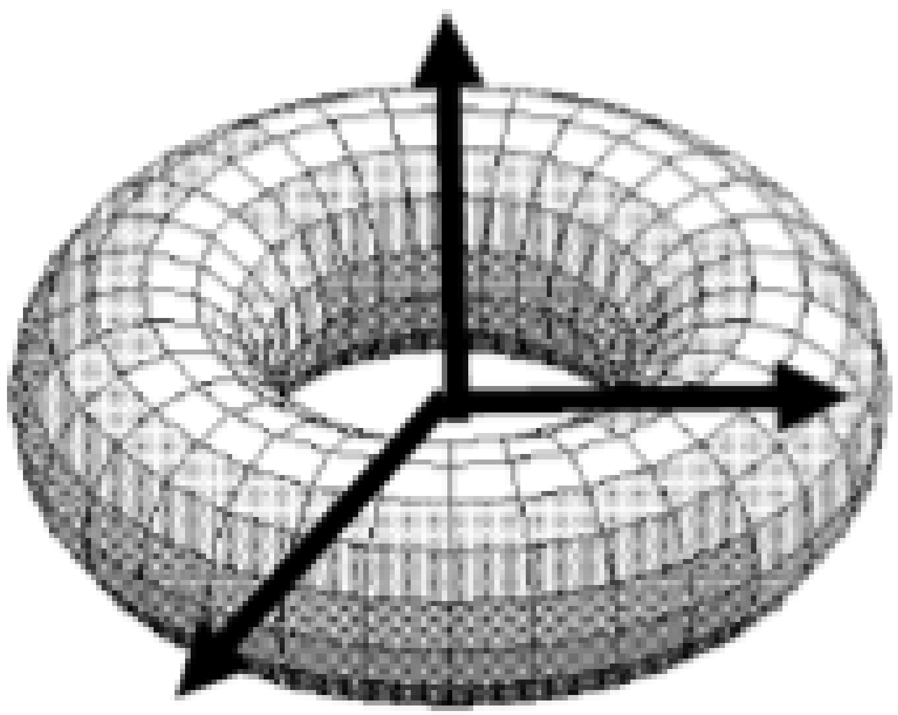
- 2D transformations
- 3D transformations

View transformation

Placing objects

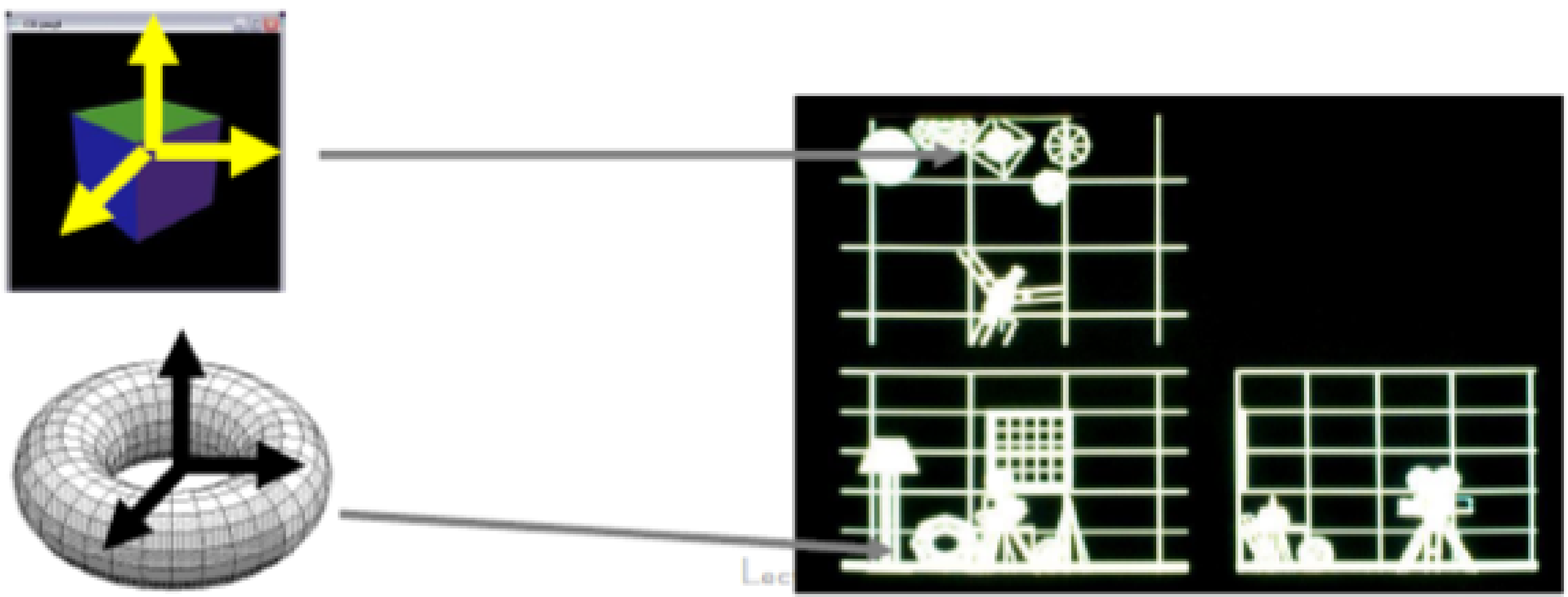
- Having prepared objects, they need to be placed in the environment
- This involves knowing the vertex coordinates of the object in the global coordinate system
- Object vertex coordinates are only defined locally



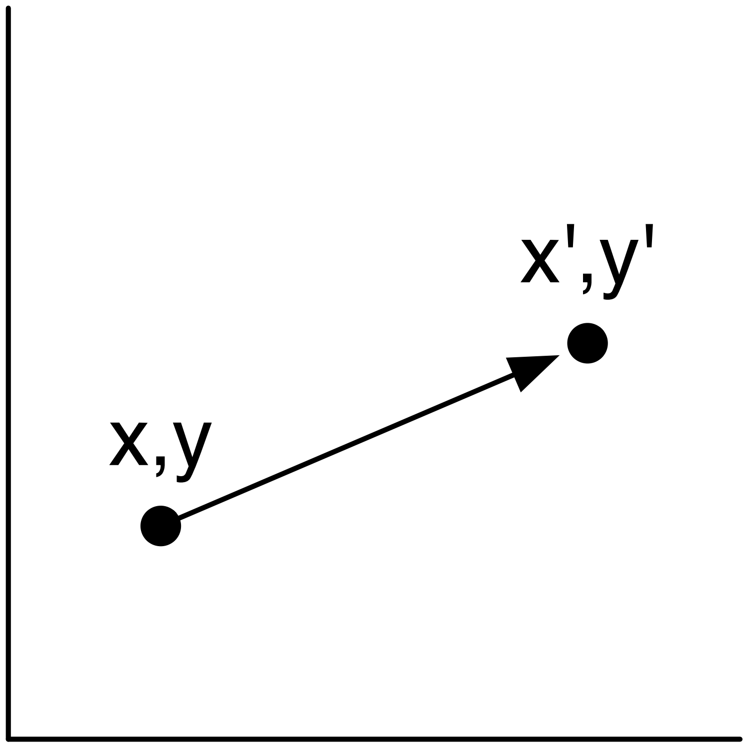


Transformations

The coordinates of the vertices of the object and translated, rotated and scaled to place them in the global coordinate system.

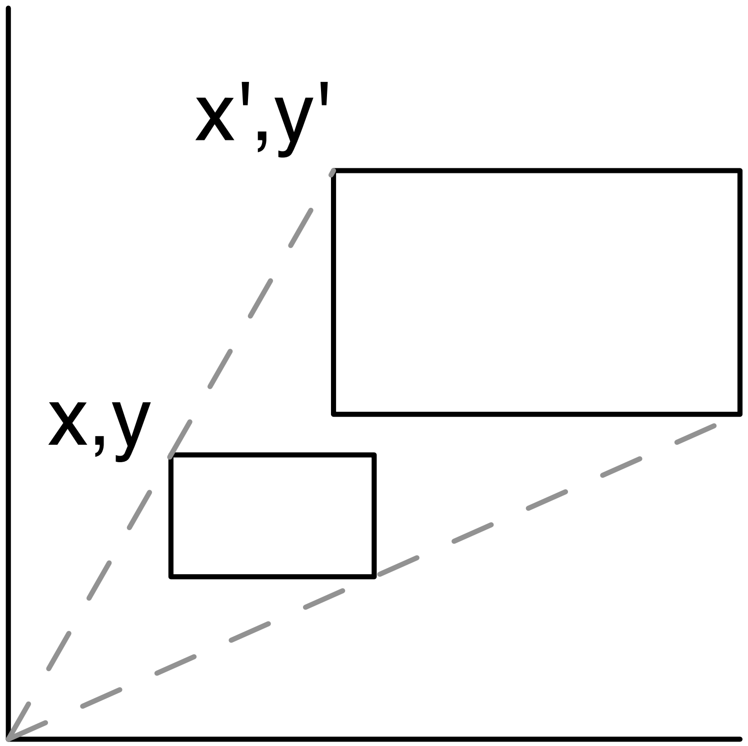


Translation



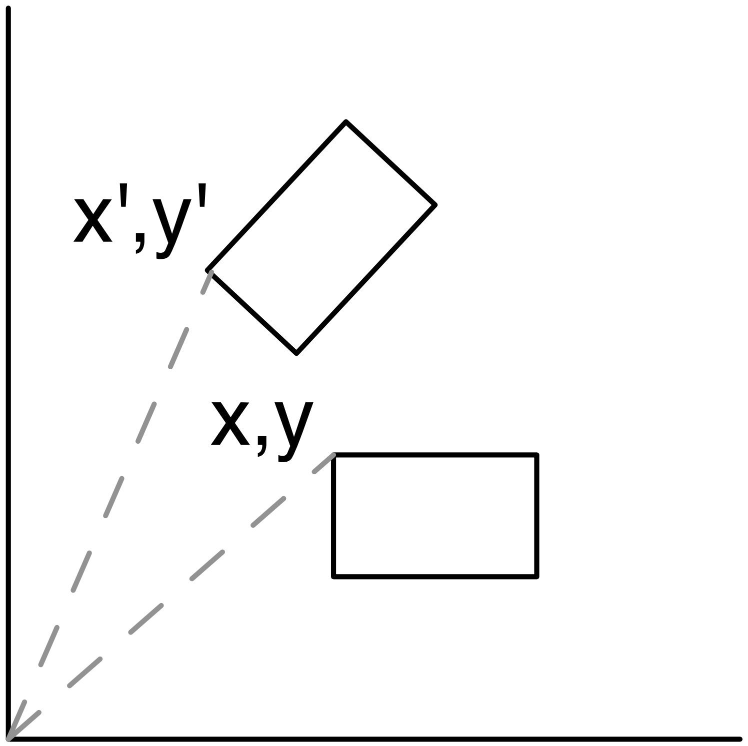
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

Scaling



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation

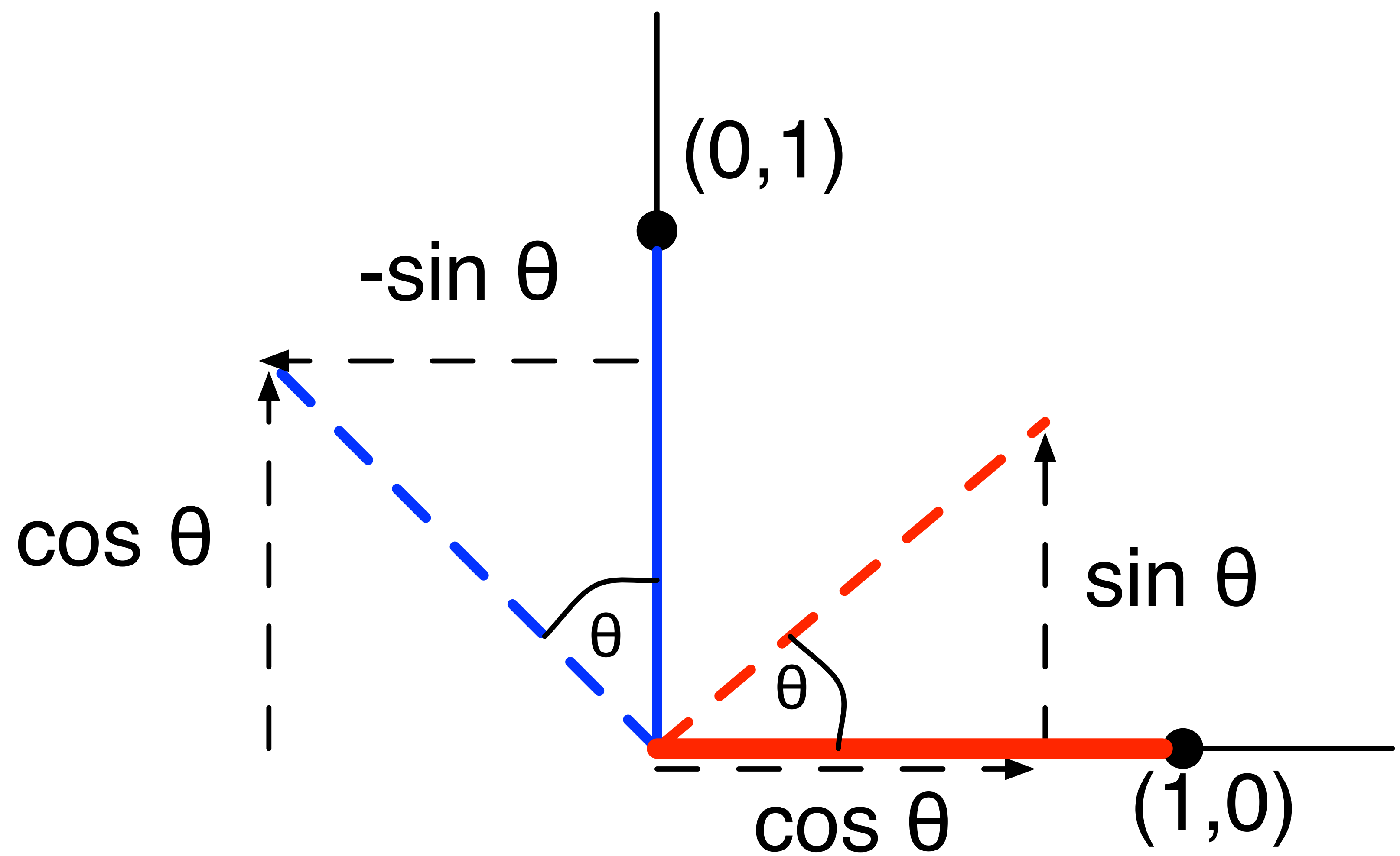


$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation

Columns of the matrix correspond to the new x and y axes after transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Operations

Translation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix}$

Scaling $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Rotation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Unfortunately these aren't all the same operation!

Homogeneous coordinates

Extend the coordinate system with a mapping back to 2D:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}$$

Then:

$$\text{Translation} \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\text{Scaling} \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

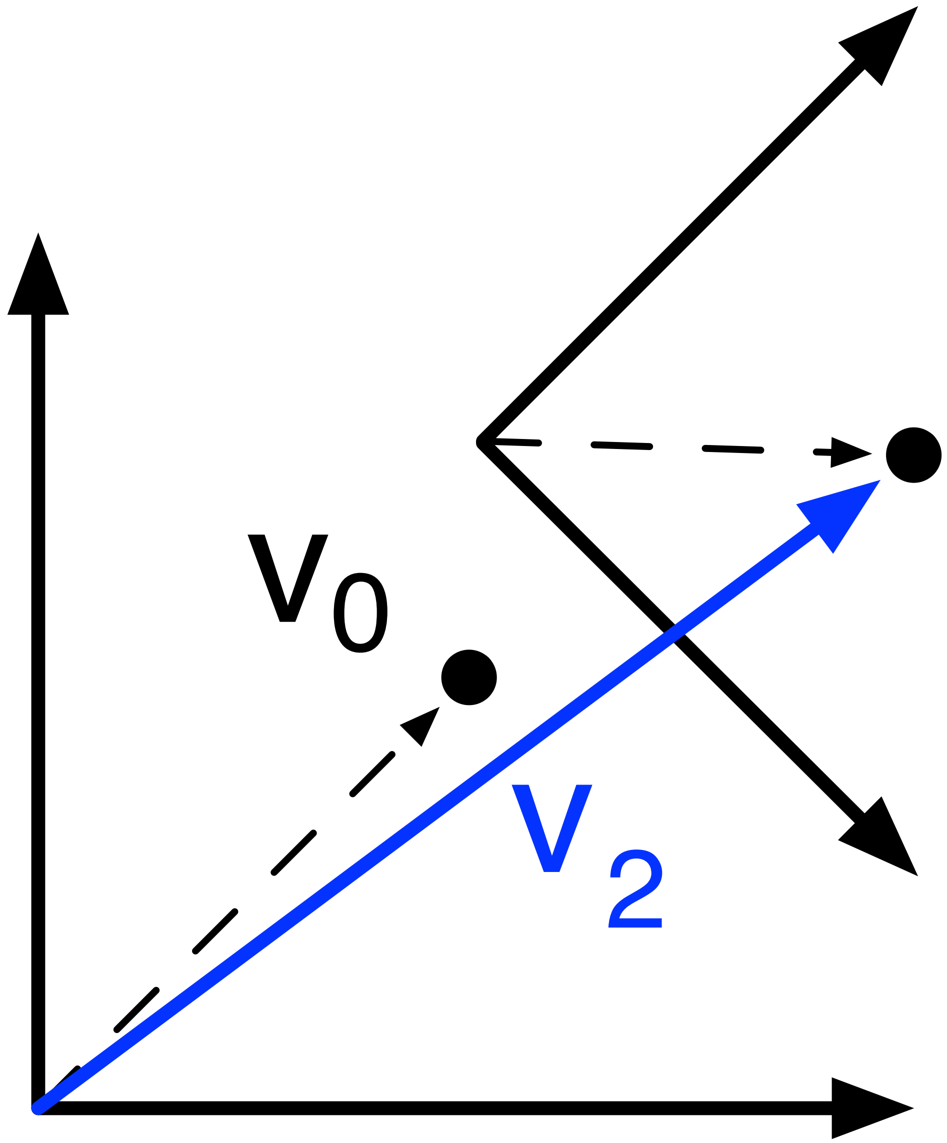
$$\text{Rotation} \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Combining transformations

- With a set of transformation matrices T, R, S, apply transformations with respect to global coordinate system: order them right to left
- E.g. Rotation R, then scaling S, then translation T, would be TSR
- Can combine these matrices into a single matrix by applying matrix multiplication.
- Note there is only one way to apply matrix multiplication, and it is not commutative (i.e. TSR!=RST)

Combining transformations

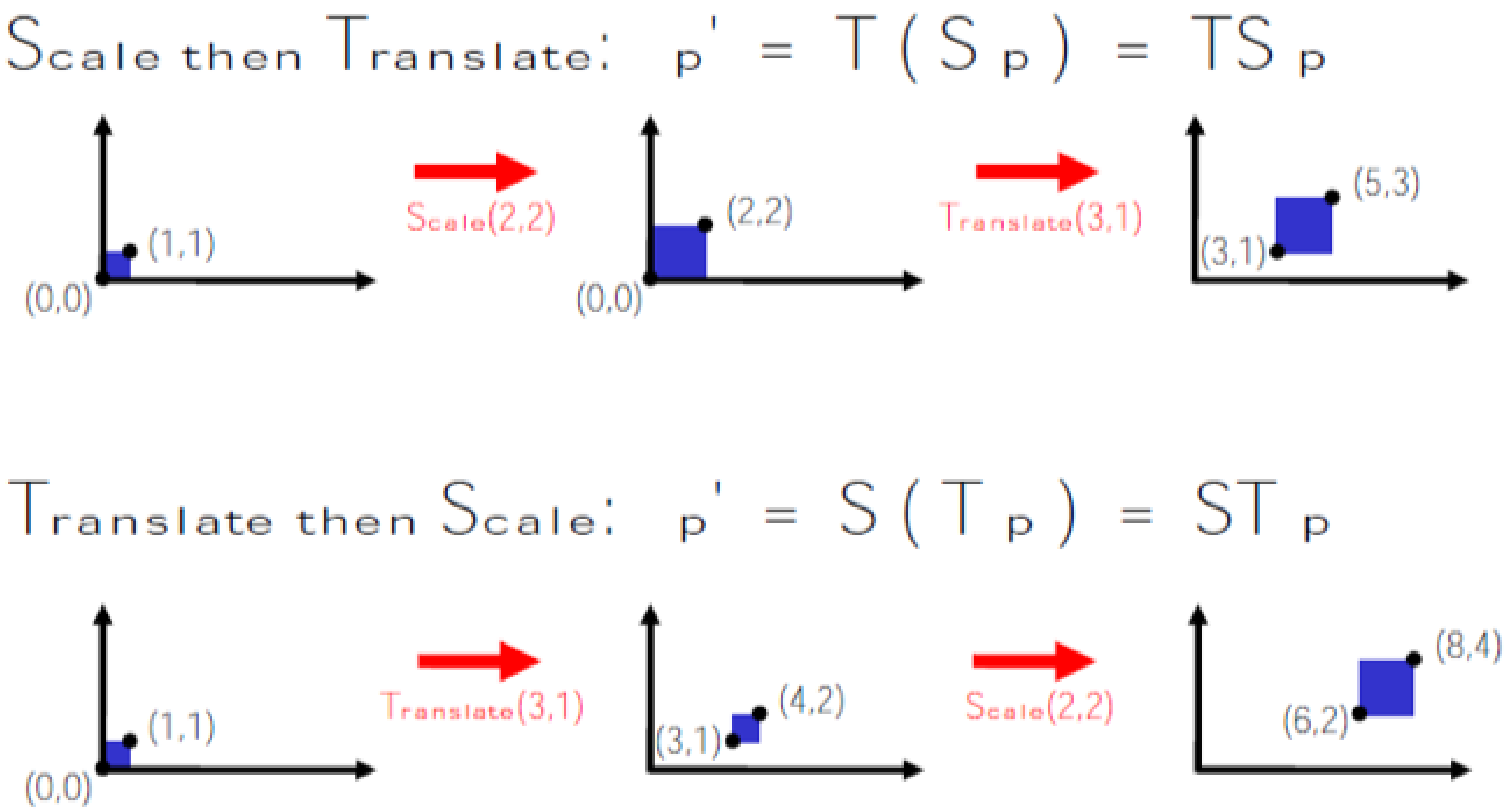
$$
\begin{aligned}
v_1 &= M_0 v_0 \\
v_2 &= M_1 v_1 \\
&\Downarrow \\
v_2 &= M_1 M_0 v_0
\end{aligned}
$$



Sequentially multiply the matrices

- Remember that matrix operations are not commutative

Example 1



Scale then Translate: p' = T (S p) = TS p
(1,1)
(0,0)
Scale(2,2)
(2,2)
(0,0)
Translate(3,1)
(5,3)
(3,1)
Translate then Scale: p' = S (T p) = ST p
(1,1)
(0,0)
Translate(3,1)
(4,2)
(3,1)
Scale(2,2)
(8,4)
(6,2)

Example 1

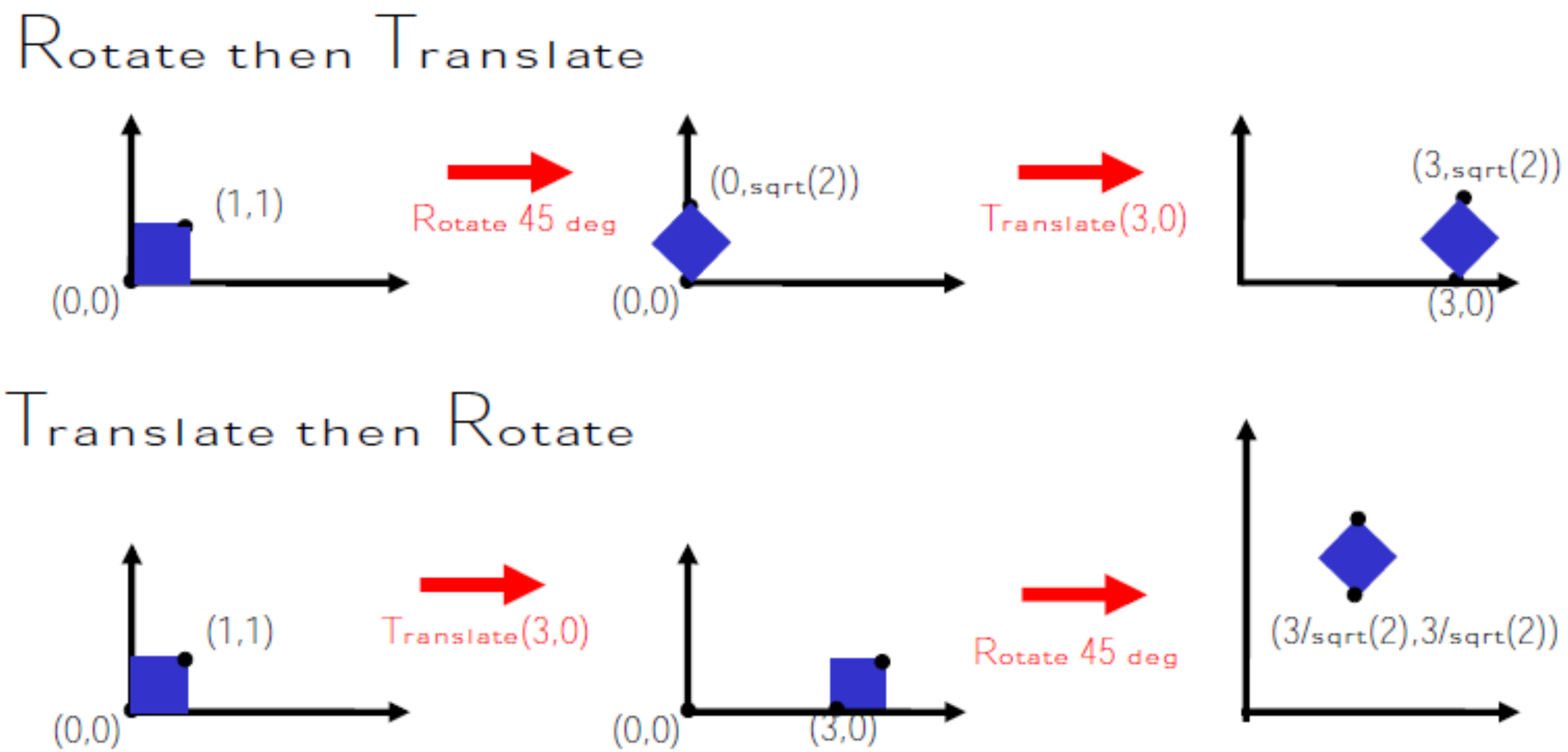
Scale then Translate: $p' = T(Sp) = TSp$

$$TS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate then Scale: $p' = S(Tp) = STp$

$$ST = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

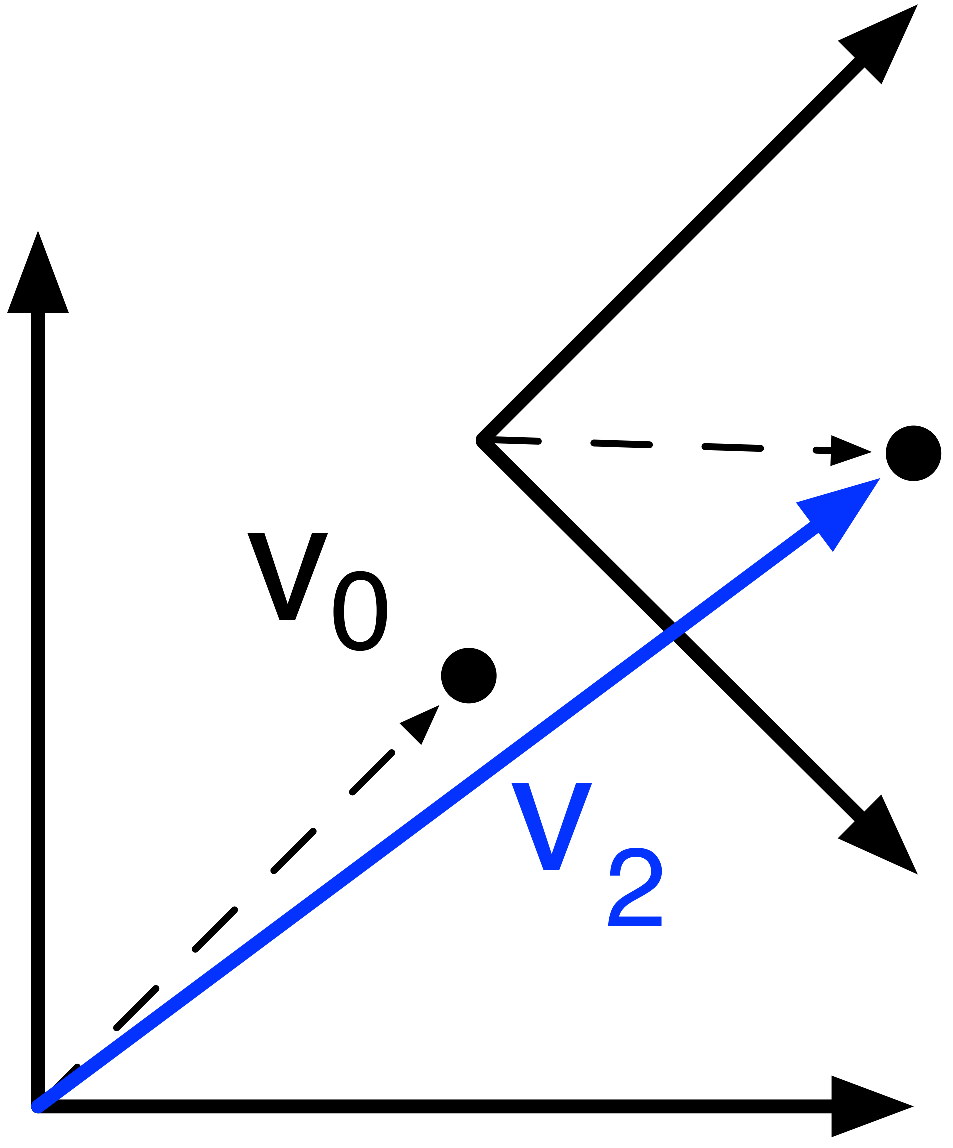
Example 2


Rotate then Translate
(1,1)
(0,0)
Rotate 45 deg
(0,sqrt(2))
(0,0)
Translate(3,0)
(3,sqrt(2))
(3,0)
Translate then Rotate
(1,1)
(0,0)
Translate(3,0)
(0,0)
(3,0)
Rotate 45 deg
(3/sqrt(2),3/sqrt(2))

A common matrix

Rotation followed by translation is often used, this is worth remembering:

$$\begin{pmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{pmatrix}$$



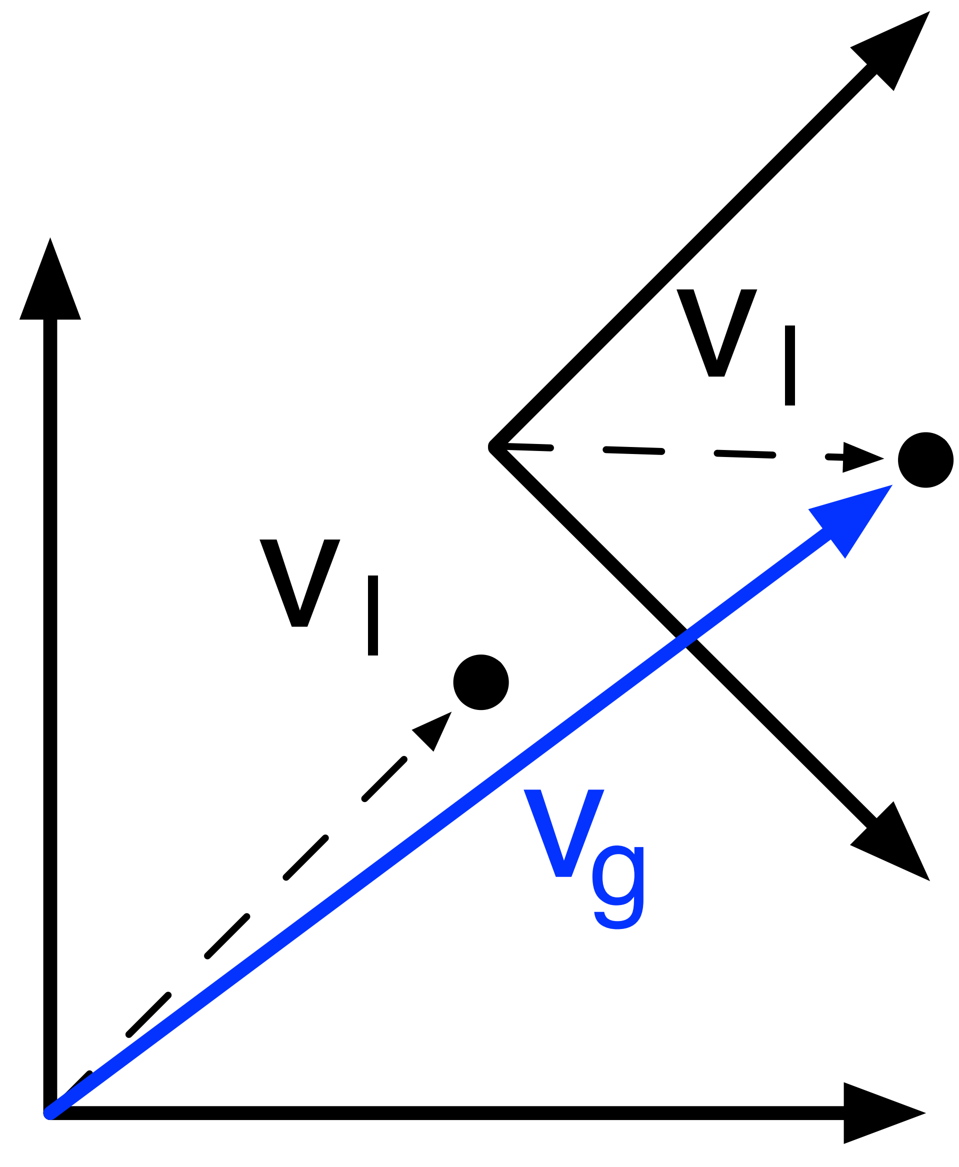
Transforming between coordinate systems

- Positioning objects (defined in a local coordinate system) in a scene
- Finding coordinates of a point with respect to a specific origin and x/y axes (calculating camera coordinates of objects)

Transformation between coordinate systems

We can interpret the transformation matrix as converting the coordinates of vertices between different coordinate systems.

$$
\begin{aligned}
v_g &= M v_l \\
v_l &= M^{-1} v_g
\end{aligned}
$$

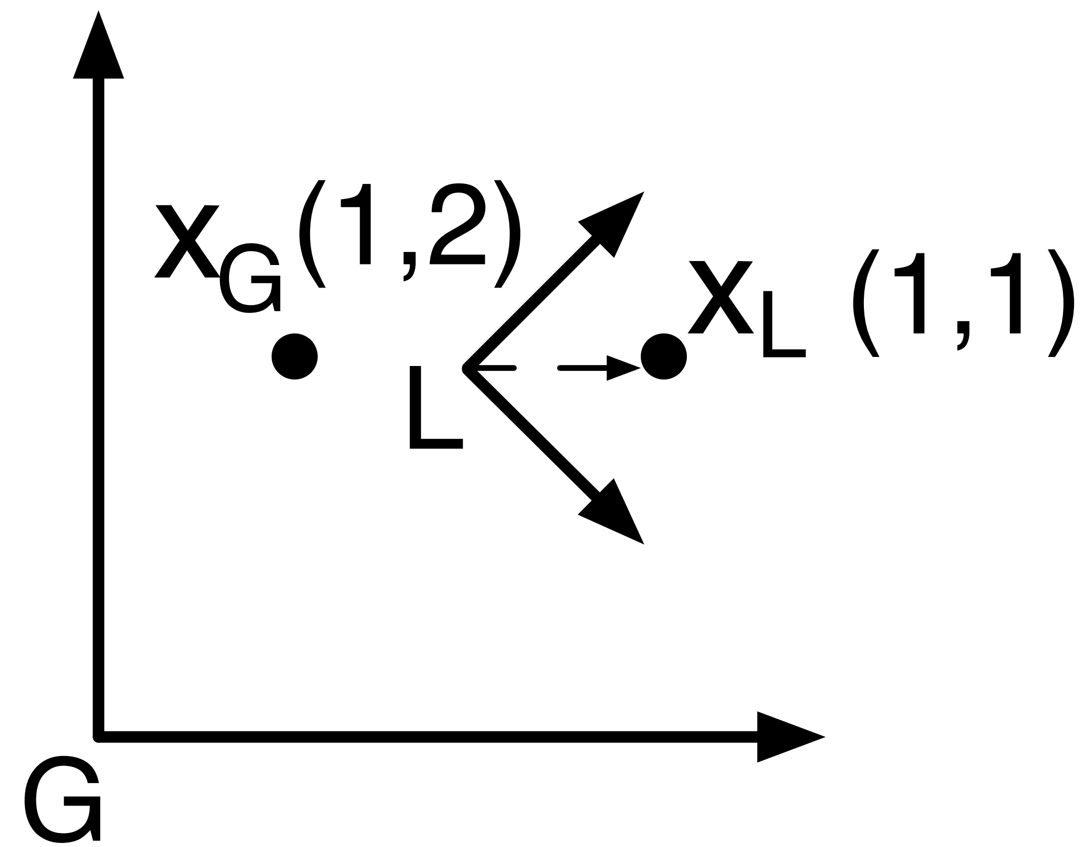


Example

- What is the position of X_G in coordinate system L?
- What is the position of X_L in coordinate system G?

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

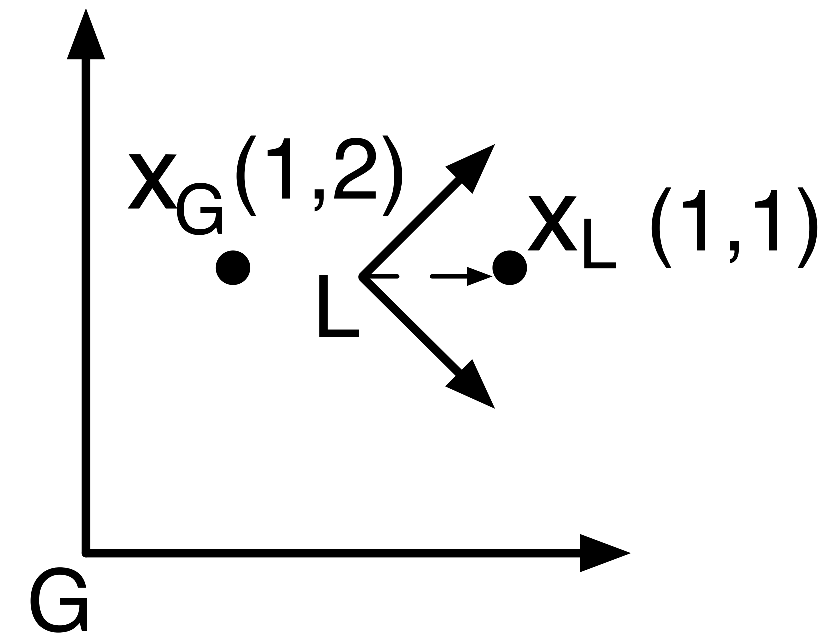
$$M^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$



Example

What is the position of X_G in coordinate system L?

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$



What is the position of X_L in coordinate system G?

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ 2 \\ 1 \end{pmatrix}$$

Transformation between coordinate systems

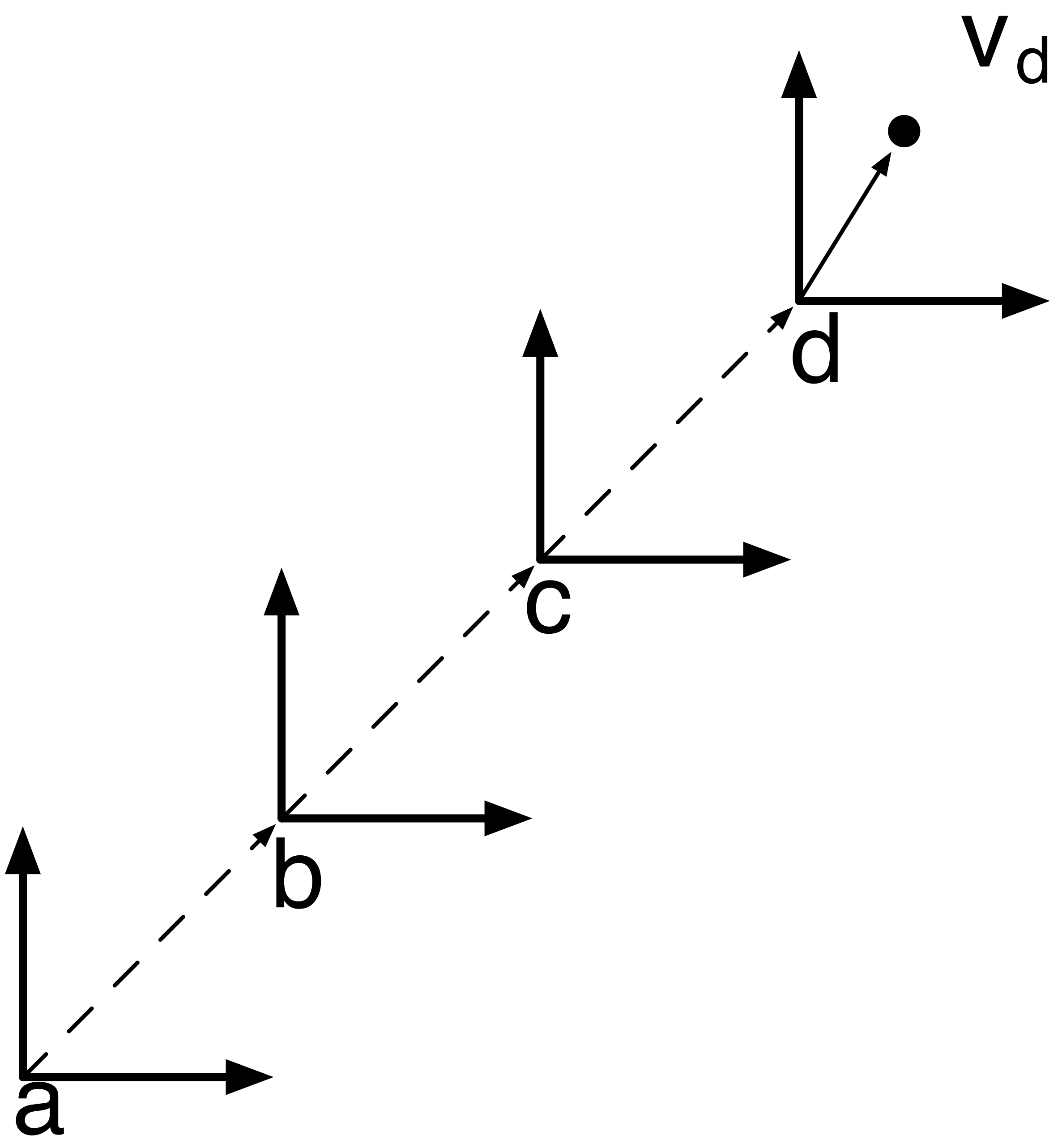
We can generalize the idea to multiple coordinate systems rather than local and global:

$$
\begin{aligned}
v_i &= M_{i \leftarrow j} v_j \\
v_j &= M_{j \leftarrow i} v_i \\
&= M_{i \leftarrow j}^{-1} v_i
\end{aligned}
$$

If we have multiple coordinate systems a, b, c and d, along with transformations $M_{a \leftarrow b}$, $M_{b \leftarrow c}$, $M_{c \leftarrow d} \ldots$

Multiple coordinate systems

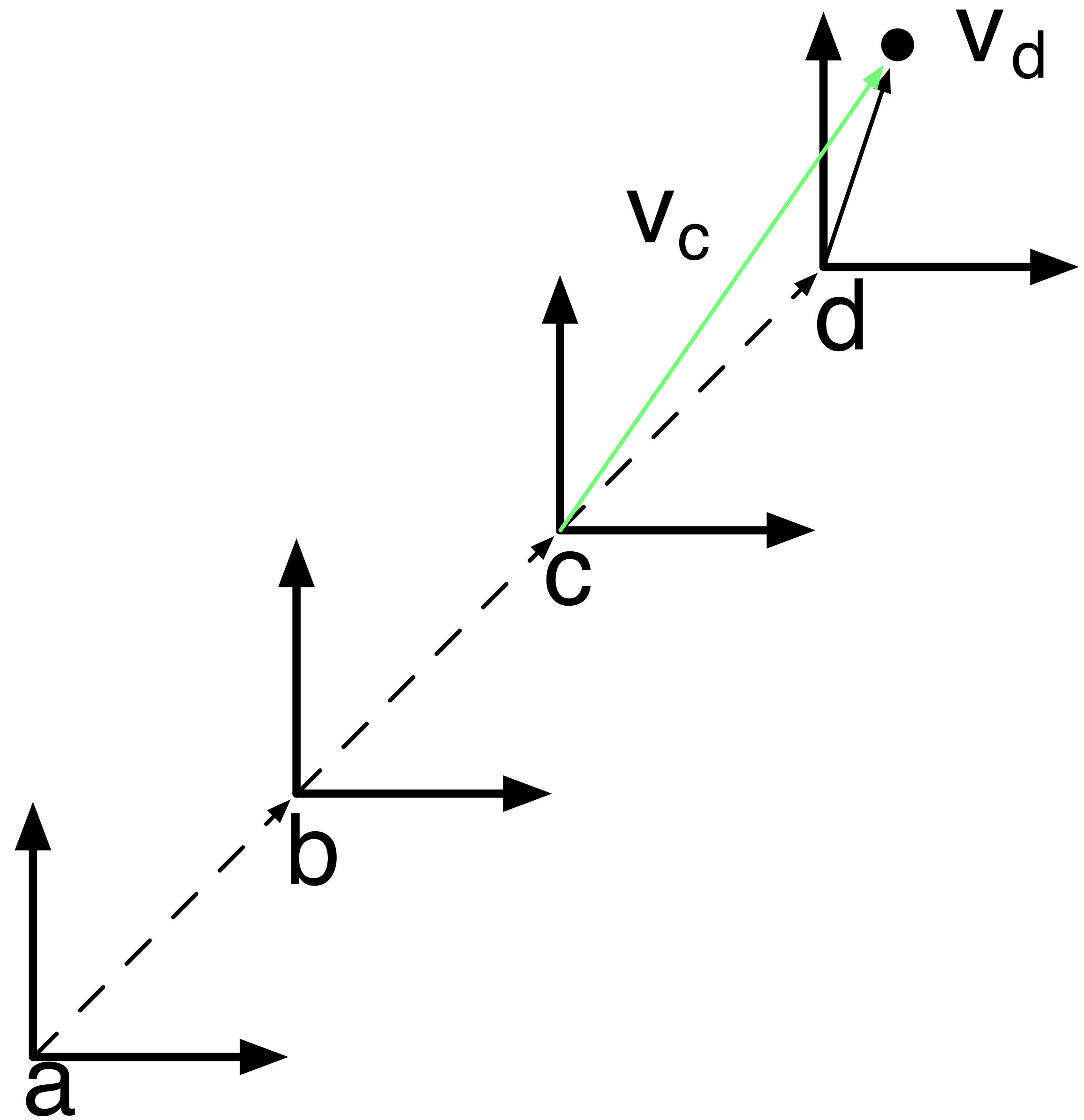
What is the position of V_d with respect to coordinate system c?



Multiple coordinate systems

What is the position of V_d with respect to coordinate system c?

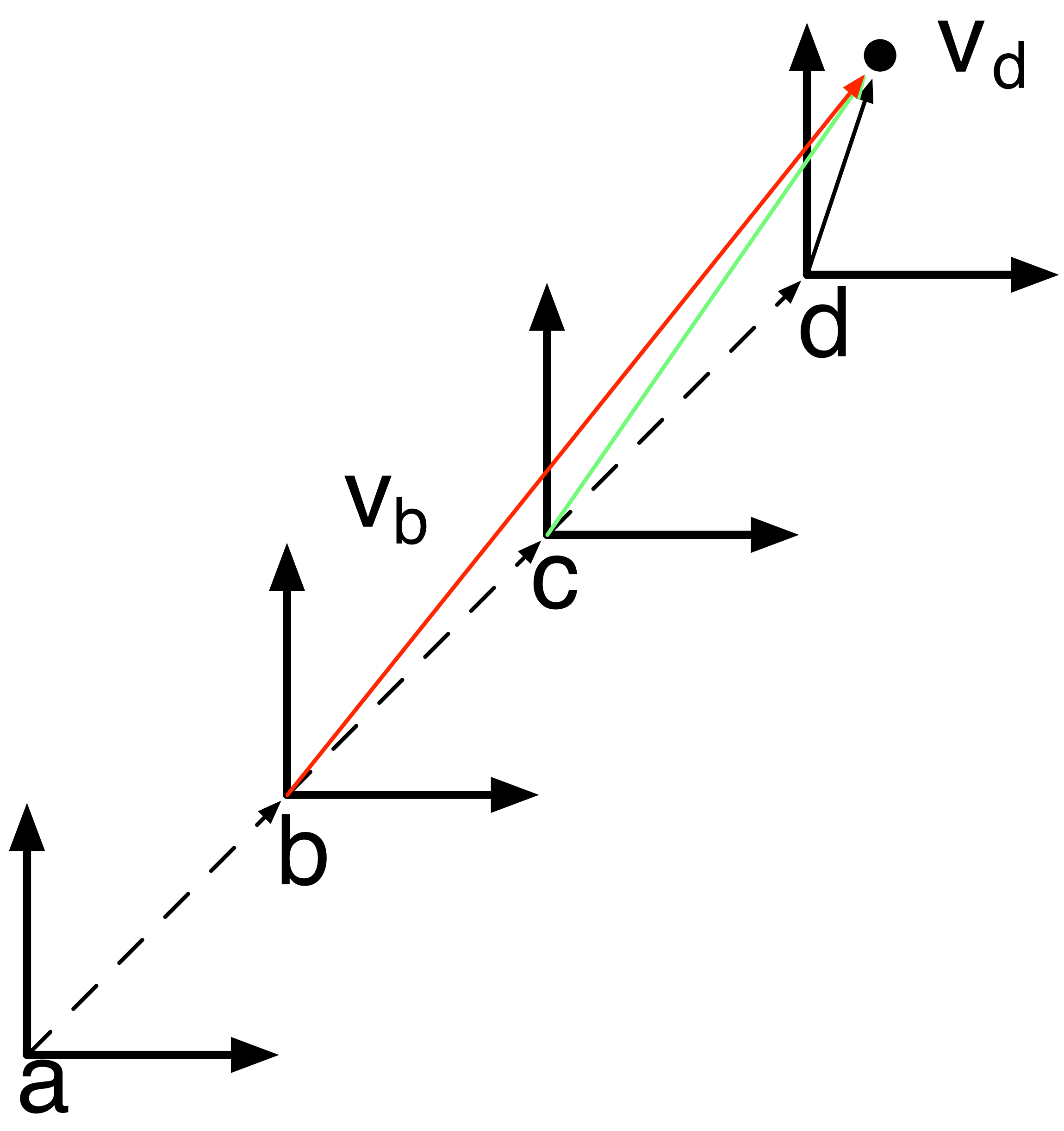
- $v_c = M_{c \leftarrow d} v_d$



Multiple coordinate systems

What is the position of V_d with respect to coordinate system b?

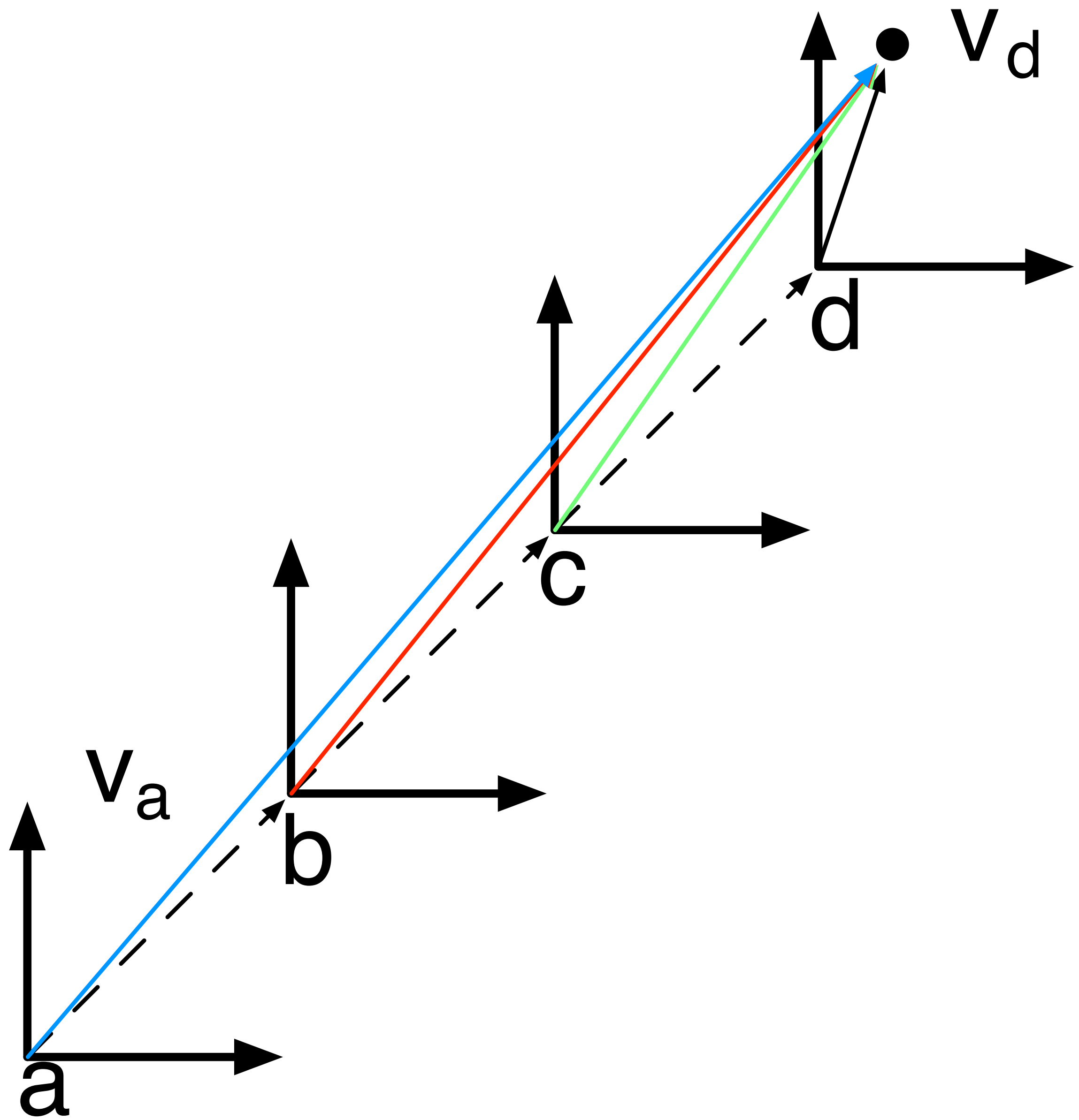
- $v_c = M_{c \leftarrow d} v_d$
- $v_b = M_{b \leftarrow c} v_c = M_{b \leftarrow c} M_{c \leftarrow d} v_d$



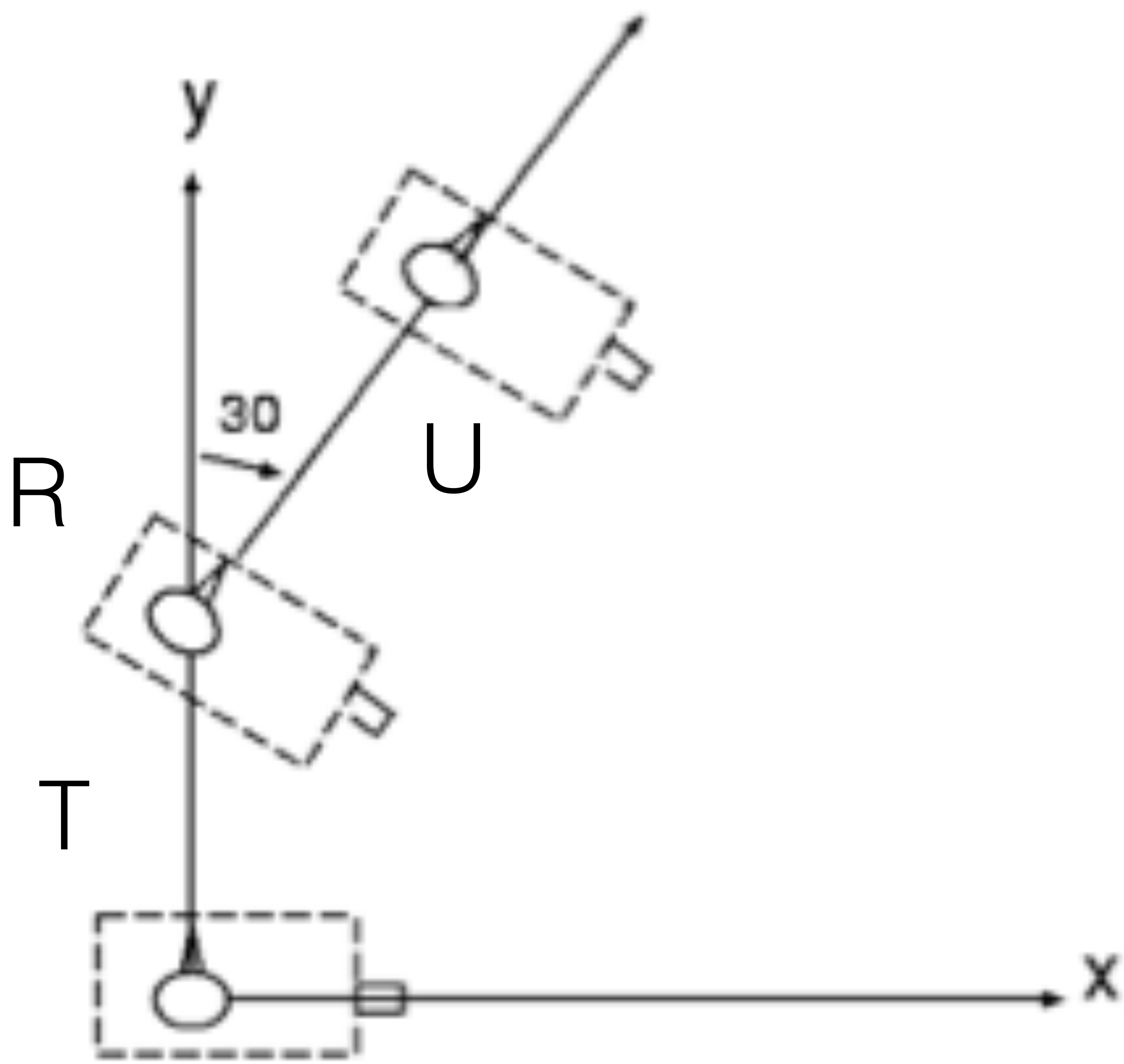
Multiple coordinate systems

What is the position of V_d with respect to coordinate system a?

- $v_c = M_{c\leftarrow d} v_d$
- $v_b = M_{b\leftarrow c} v_c = M_{b\leftarrow c} M_{c\leftarrow d} v_d$
- $v_a = M_{a\leftarrow b} v_b = M_{a\leftarrow b} M_{b\leftarrow c} M_{c\leftarrow d} v_d$



Combining transformations

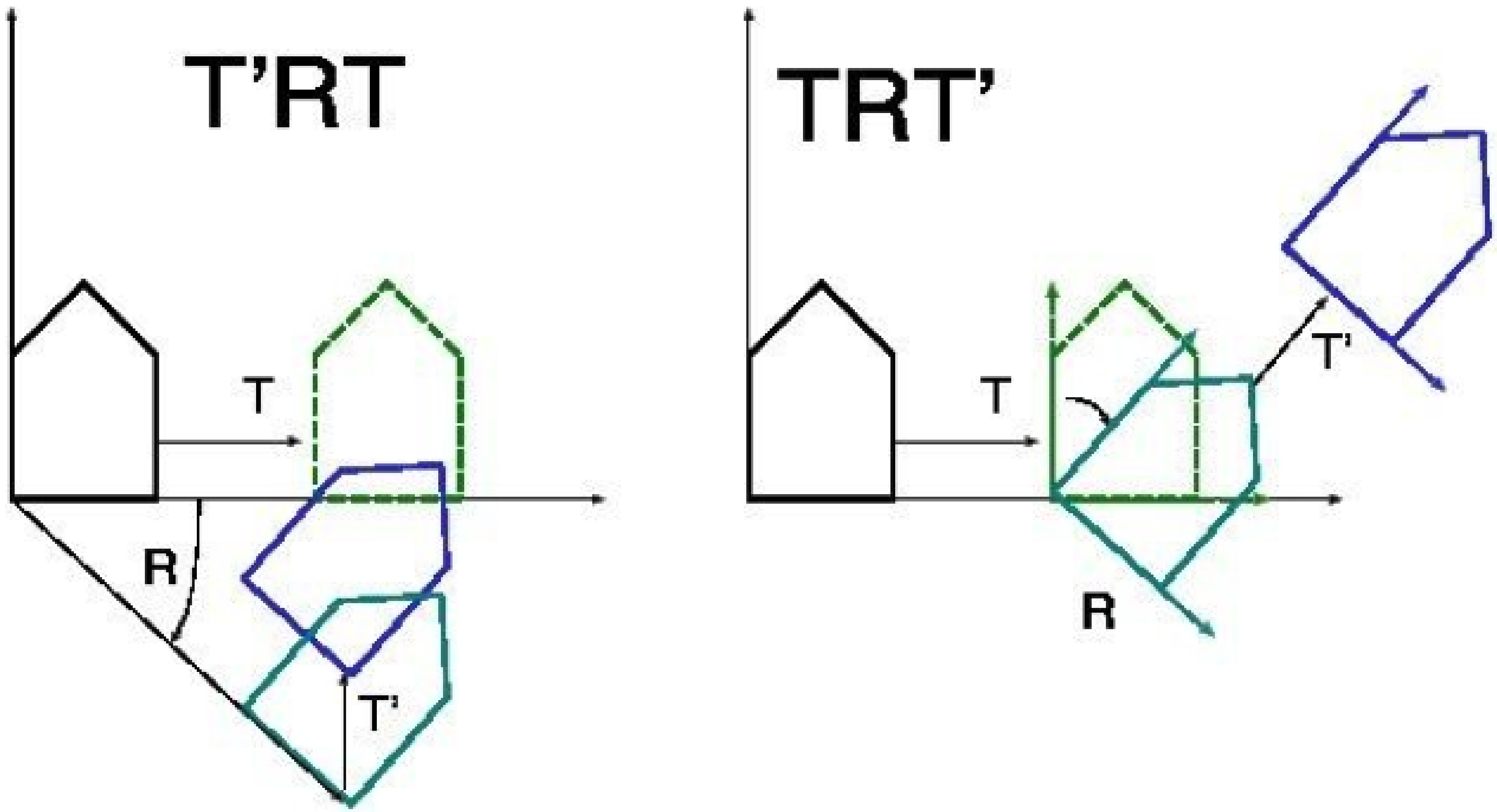


- Less common: *special case* where we want to apply the transformations in our matrices with respect to the *local* coordinate system as it moves. Matrices are ordered from left to right.

- E.g. a car driving forwards, turning and then driving forwards again. The second transformation forwards should be along the local y axis of the car, not of the global coordinate system. Apply matrices as TRU

- This is equivalent to performing U then R then T in the global coordinate system - just an easier way of constructing it.

Order of multiplication

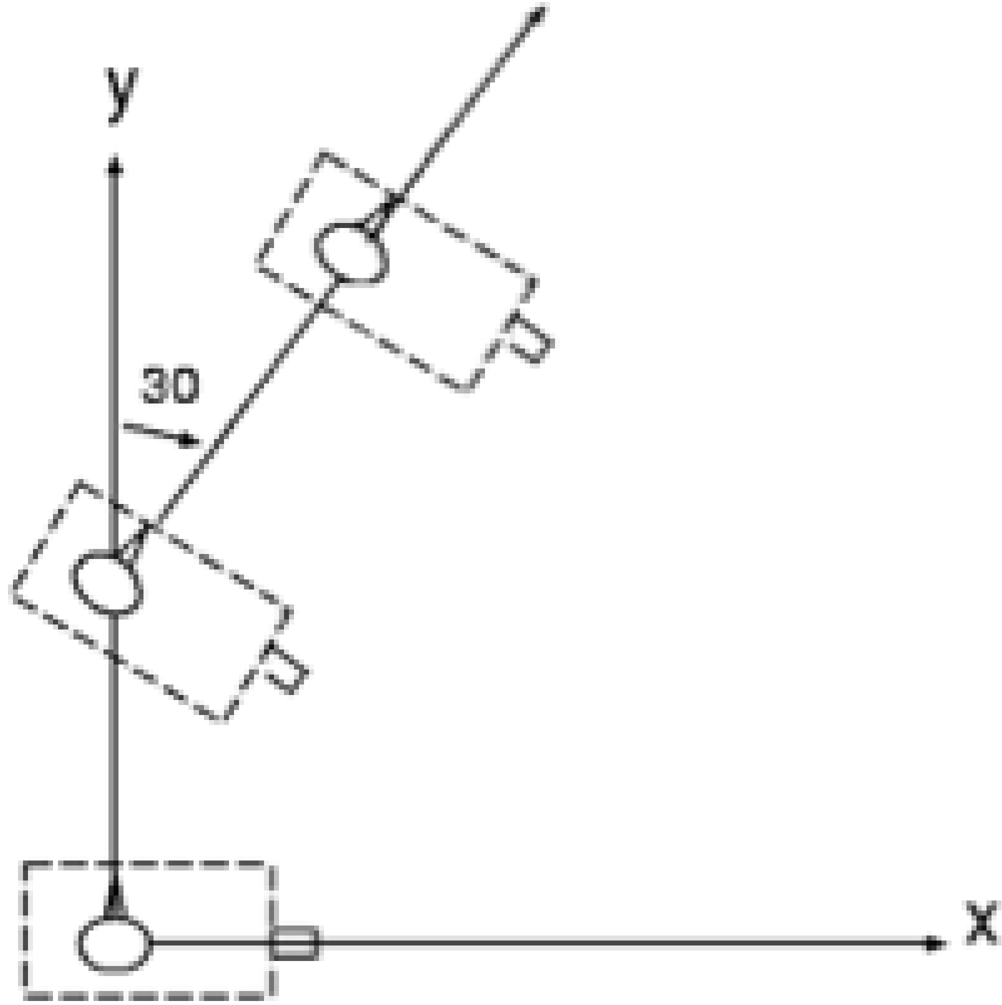


Example - driving a car

Sitting in a car with the mirror $0.4m$ to your right, you

- drive $5m$ forwards
- turn 30 degrees right
- drive $2m$ forwards

Where is the side mirror relative to its original position?



Example - driving a car

The transformation matrix of driving forwards $5m$ is:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

The transformation matrix of turning 30 degrees to the right is:

$$R = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transformation matrix of driving forwards $2m$ is:

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution

$$x' = T_1 R T_2 x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0 \\ 1 \end{pmatrix}$$

$$x' = \begin{pmatrix} 0.65 \\ 6.53 \\ 1 \end{pmatrix}$$

3D translation

Very simple to extend 2D case to 3D:

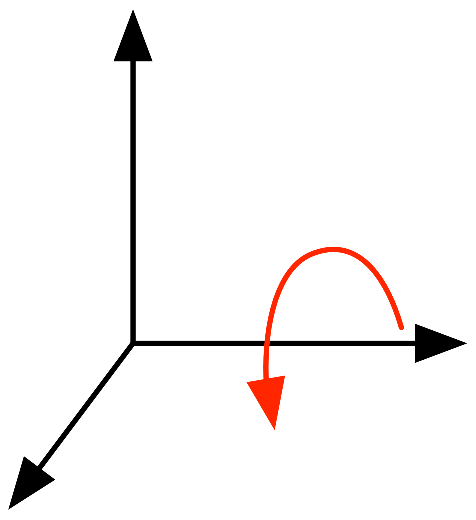
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D scaling

Very simple to extend 2D case to 3D:

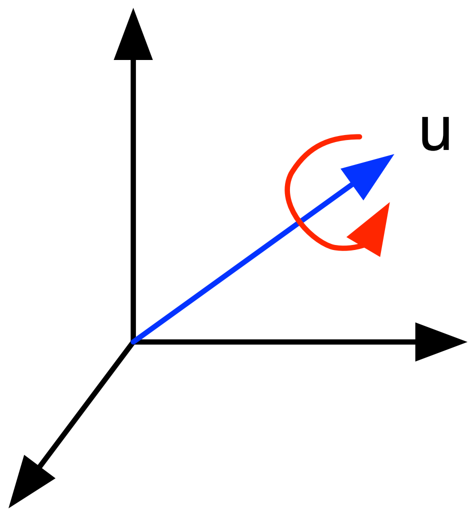
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D rotation - X axis



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D rotation - arbitrary axis



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} u_x^2 + c(1-u_x^2) & u_x u_y(1-c) - u_z s & u_z u_x(1-c) + u_y s & 0 \\ u_x u_y(1-c) + u_z s & u_y^2 + c(1-u_y^2) & u_y u_z(1-c) - u_x s & 0 \\ u_z u_x(1-c) - u_y s & u_y u_z(1-c) + u_x s & u_z^2 + c(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

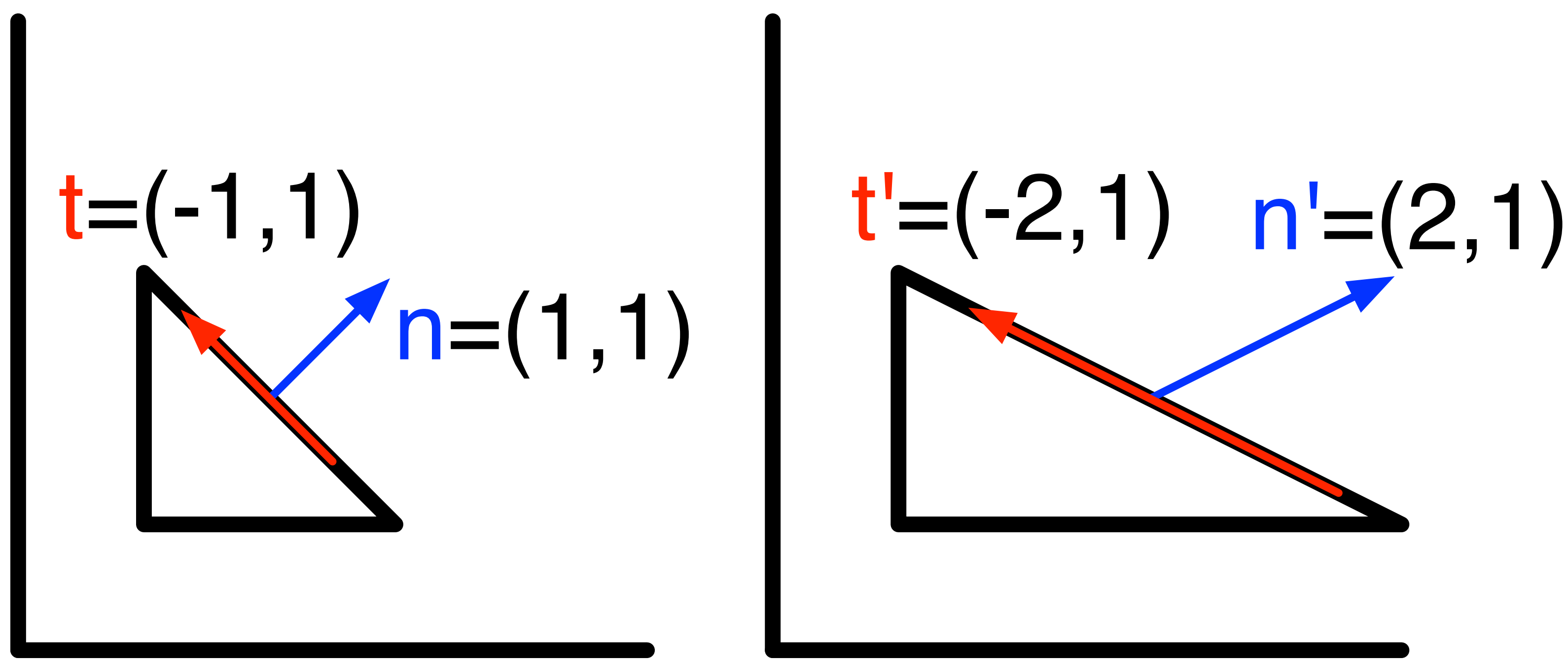
$c = \cos\theta, \; s = \sin\theta$

Transforming vertex normals

Transforming vertex normals, we need to take care to ensure they remain perpendicular to the surface.

Tangent vectors remain tangent to the surface after transformation.

For tangent vector t and normal vector n transformed by scaling S:



$$S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad n' = Sn,\ t' = St.$$

Transforming vertex normals

Transforming vertex normals, we need to take care to ensure they remain perpendicular to the surface.

Tangent vectors remain tangent to the surface after transformation.

For any tangent vector t and normal vector n transformed by M:

$$\begin{aligned} n \cdot t &= 0 \\ n^T (M^{-1} M) t &= 0 \\ (n^T M^{-1})(Mt) &= 0 \end{aligned}$$

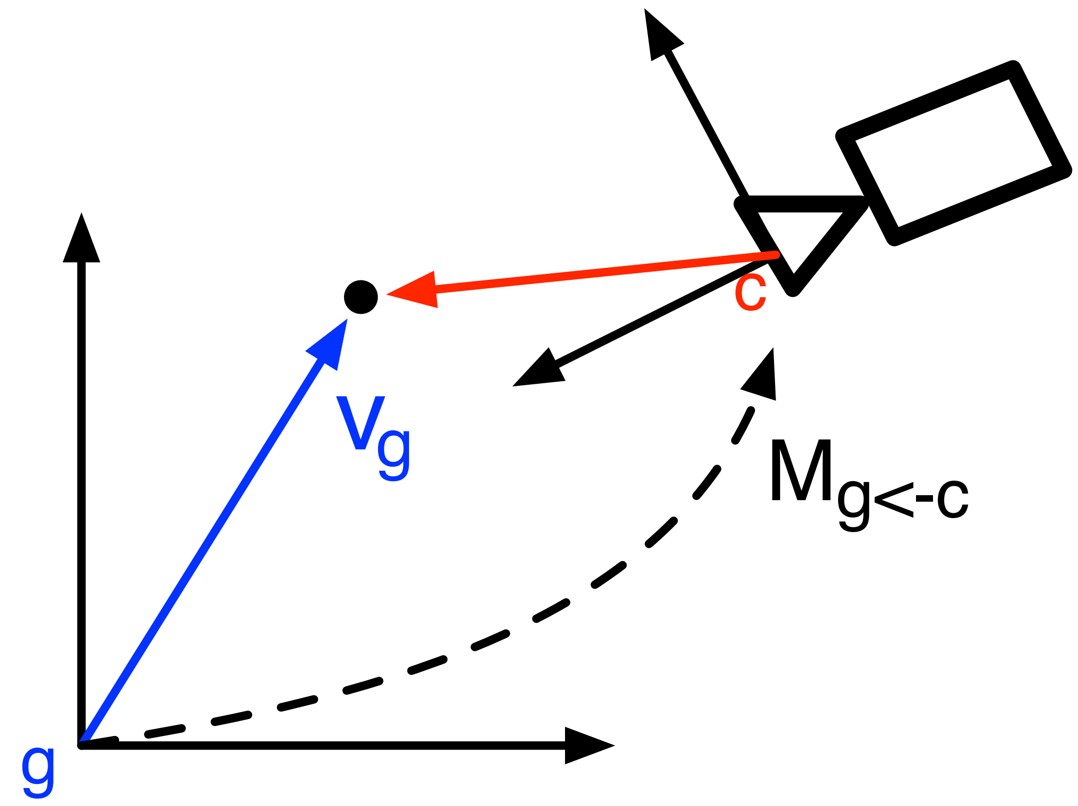
Therefore $n^T M^{-1}$ is perpendicular to the transformed Mt for any t. So the desired normal vector is:

$n' = (M^{-1})^T n$

Shirley 6.2.2

View transformation

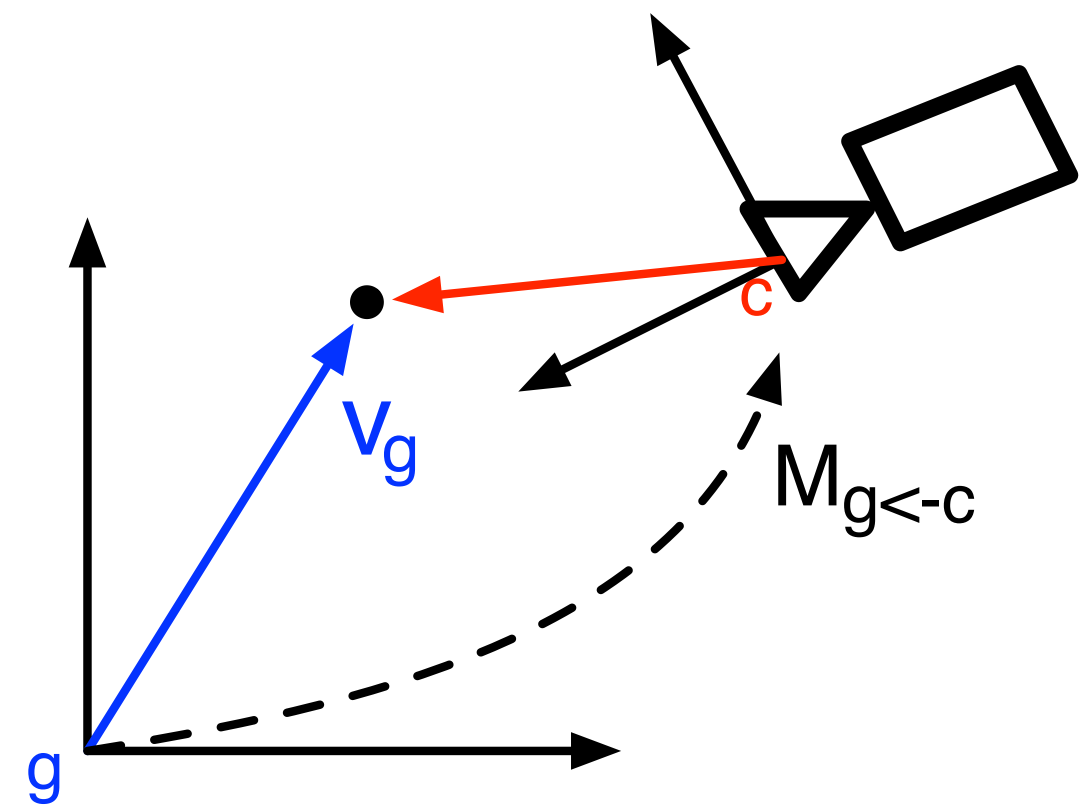
- Transform to coordinate system of camera
- Use the orientation and translation of camera from the origin of the world coordinate system



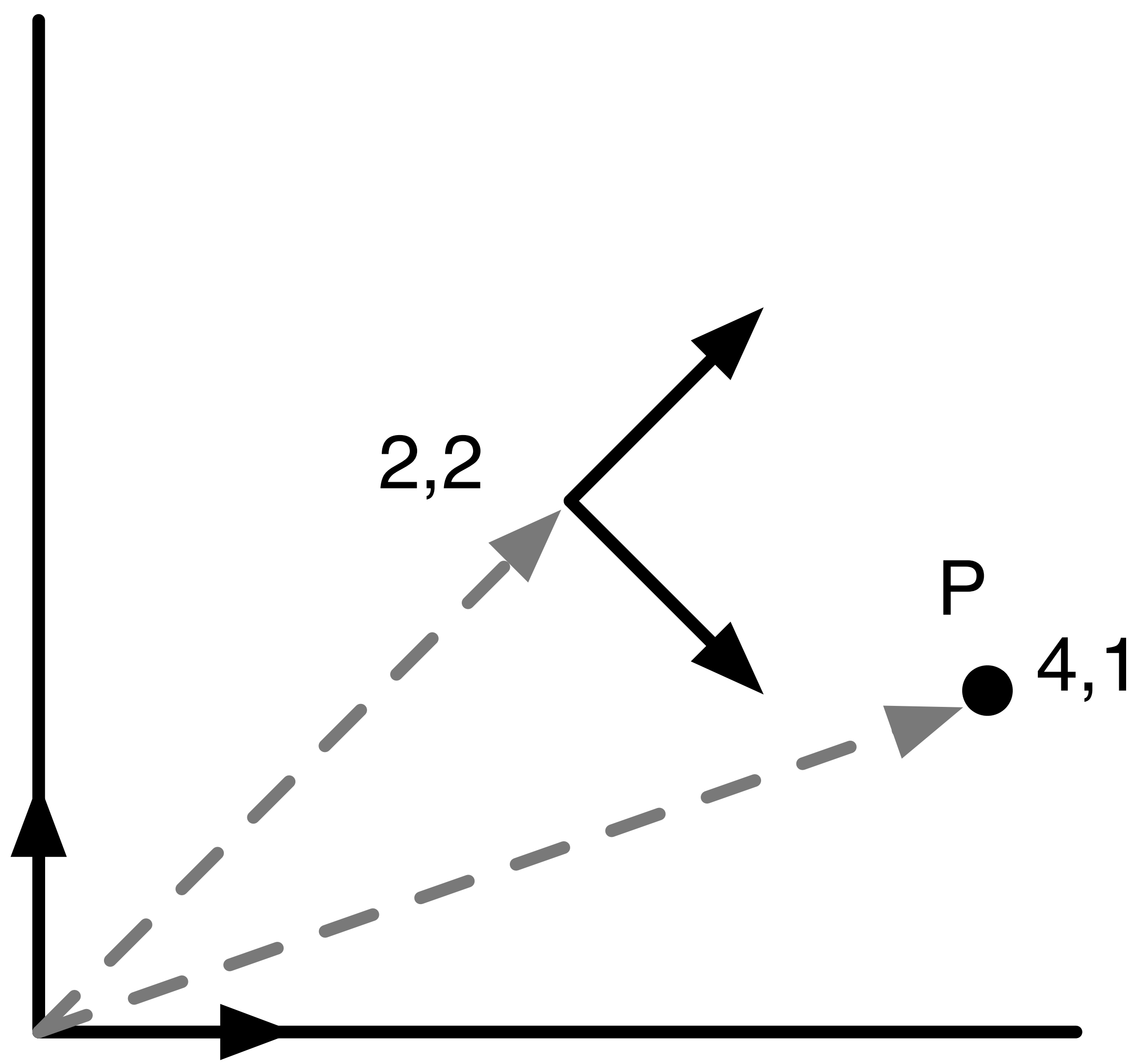
View transformation

We want to convert a vertex v_g from the global coordinate system to a vertex in the camera coordinate system v_c. We can do this using the camera to world transformation matrix $M_{g \leftarrow c}$.

$$\begin{aligned} v_g &= M_{g \leftarrow c} v_c \\ v_c &= M_{g \leftarrow c}^{-1} v_g \\ &= M_{c \leftarrow g} v_g \end{aligned}$$



Example



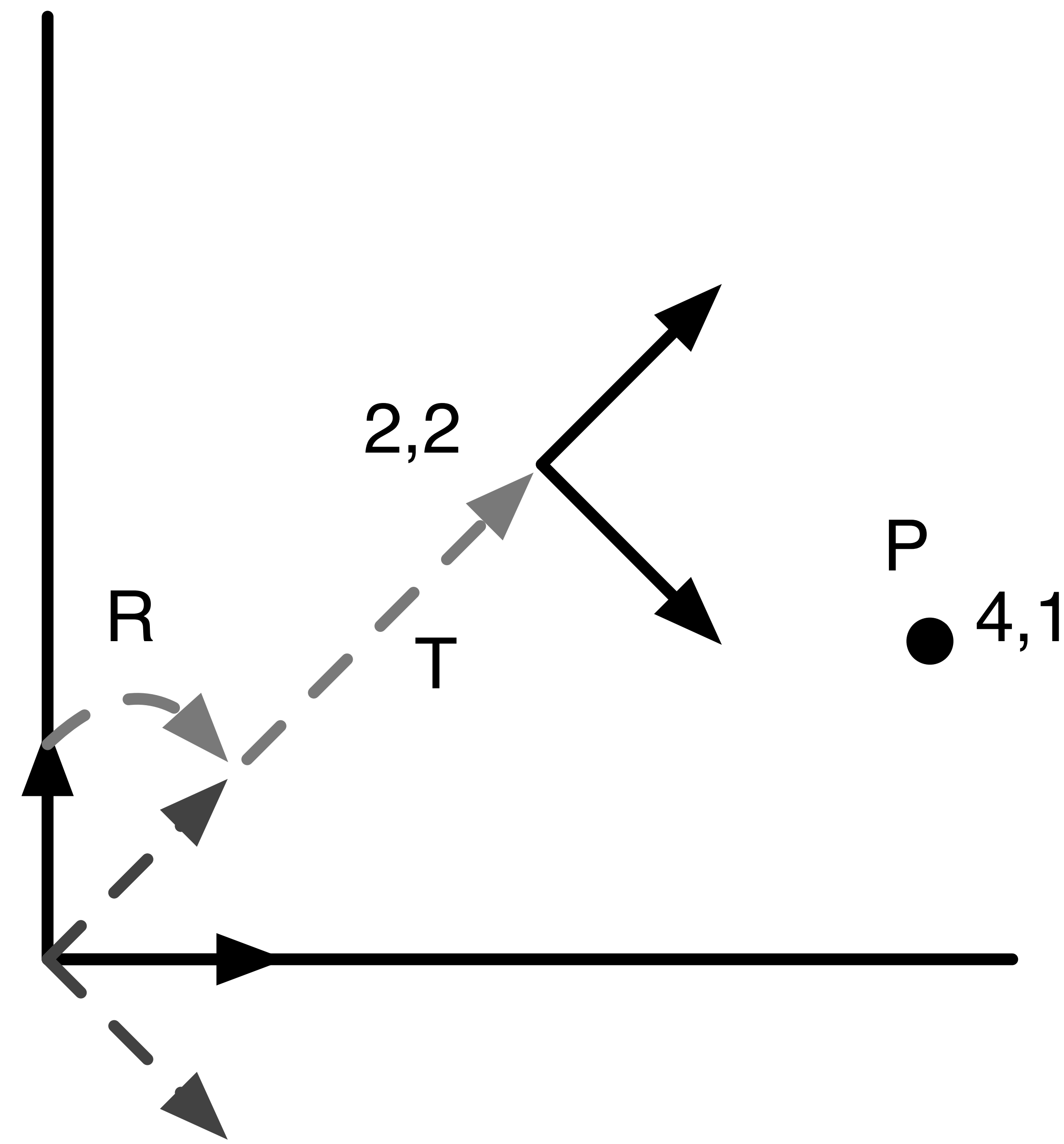
- Camera at (2,2), rotated 45 degrees clockwise
- We want to know the coordinates of P in the camera coordinate system i.e. starting from (2,2), what combination of the new x axis and y axis are needed?

Example

- Rotation then translation

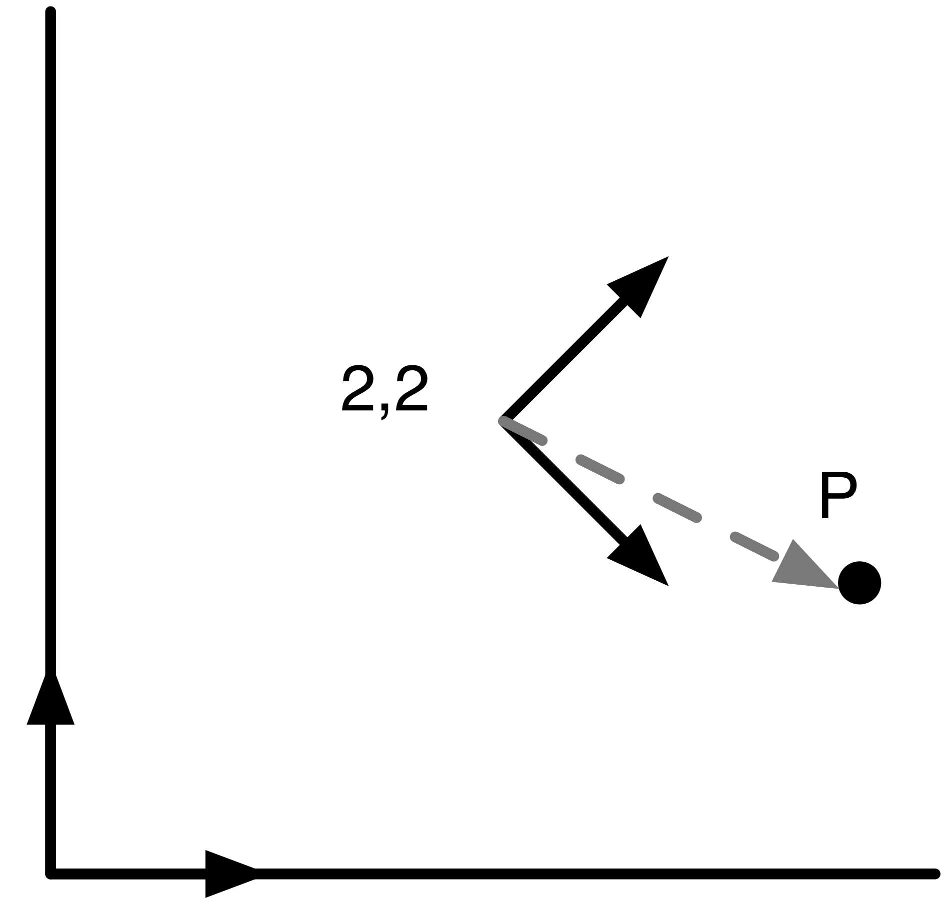
$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$



$$TR = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Example



- To find P with respect to the new x and y axes:

$$M_{l\to g} = TR = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{g\to l} = M_{l\to g}^{-1} = (TR)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$P' = M_{g\to l}P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

Summary

- Three kinds of transformation - translation, rotation and scaling
- In homogeneous coordinates, transformations can be represented by matrix multiplication
- After transforming objects to world coordinates, transform to coordinate system of the camera for viewing

References

Recommended reading:

- Shirley, Chapter 6.1 to 6.1.5 inclusive, 6.2 to 6.5 inclusive

Extra reference materials:

- Foley, Chapter 5 (Geometrical Transformations)