Computer Graphics 17 - Curves and Surfaces 2

Tom Thorne

Slides courtesy of Taku Komura
www.inf.ed.ac.uk/teaching/courses/cg

Overview

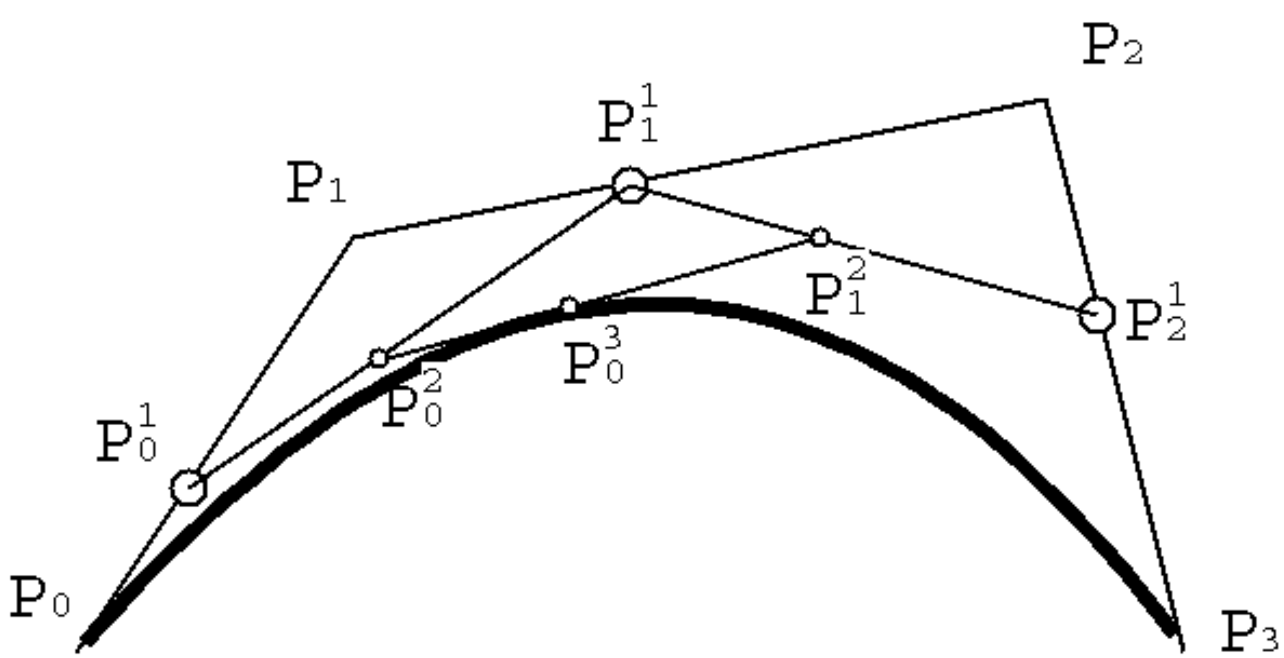
- More on Bézier and B-splines
  - **de Casteljau's algorithm**
  - General form of B-splines
  - de Boor's algorithm
  - Knot insertion
- NURBS
- Subdivision surfaces

de Casteljau's algorithm

- A method to evaluate (sample points in) or draw a Bézier curve

- Works with Bézier curves of any degree

- A precise method to evaluate the curve



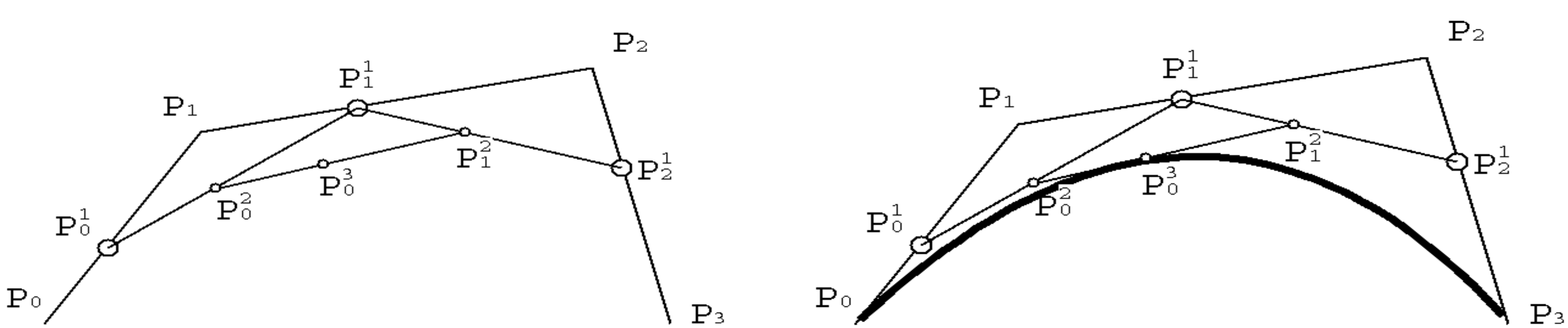
de Casteljau's algorithm

- Given the control points $P_1, \ldots, P_n$ and the parameter value $0 \leq t \leq 1$
- Repeat the following procedure:

$$P_i^r(t) = (1-t)P_i^{r-1}(t) + tP_{i+1}^{r-1}(t)$$

$$P_i^0(t) = P_i$$

- Then $P_0^n(t)$ is the point with parameter value t on the Bézier curve



Why does this work?

- In the quadratic Bézier curve case with 3 control points, P_0, P_1, P_2

$$
\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1 + tP_1^1
\end{aligned}
$$

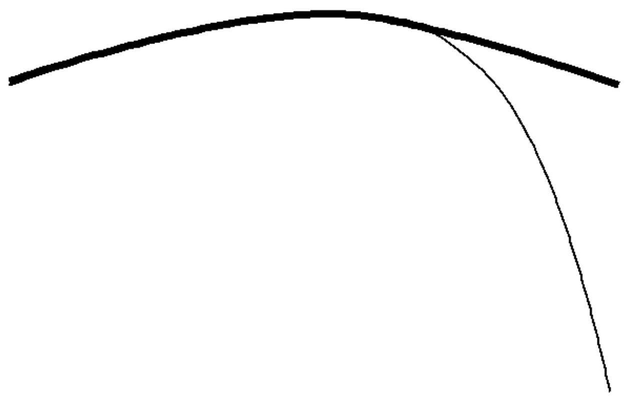
- By inserting the first two equations into the third we obtain

$$P_0^2(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

- Doing this for 4 control points will give the cubic formula we saw last week

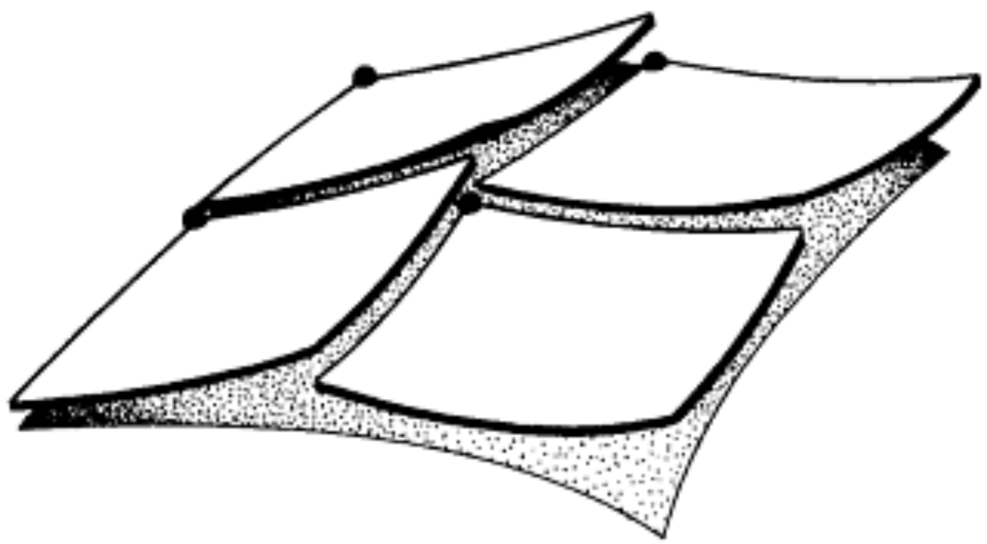
Why do we need this?

- The explicit representation can result in some instability
- Control points are randomly changed by 0.001
- The curve from de Casteljau's algorithm stays almost the same
- The curve from the polynomial basis form can deviate from the original curve if the degree is high



Connecting Bézier patches

- The same thing applies to patches
- The degree of the surface can easily become high since it is the multiplication of two curves, e.g. Bicubic is of degree 6
- The error of 16 control points is accumulated



Overview

- More on Bézier and B-splines
  - de Casteljau's algorithm
  - **General form of B-splines**
  - de Boor's algorithm
  - Knot insertion
- NURBS
- Subdivision surfaces

B-Splines: general form

a B-spline of order k (polynomial of degree k-1) is a parametric curve composed of a linear combination of basis B-splines $B_{i,k}$:

$P_i \, (i = 0, \ldots, m)$ are the control points

$$p(t) = \sum_{i=0}^{m} P_i B_{i,k}(t)$$

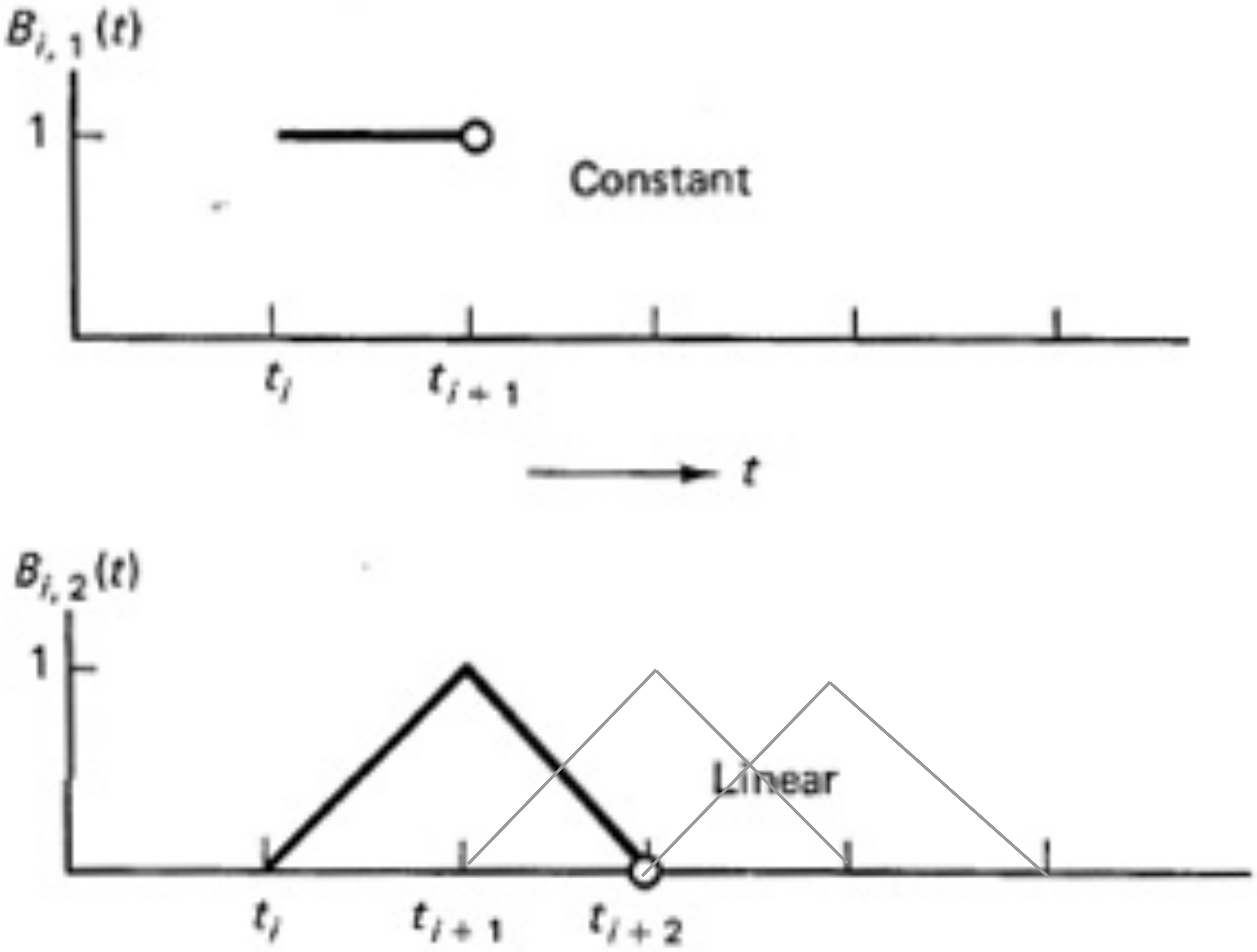
Knots: $t_0 \le t_1 \le \ldots \le t_{k+m}$ - the knots subdivide the domain of the B-spline curve into a set of knot spans $[t_i, t_{i+1})$

The B-splines can be defined by

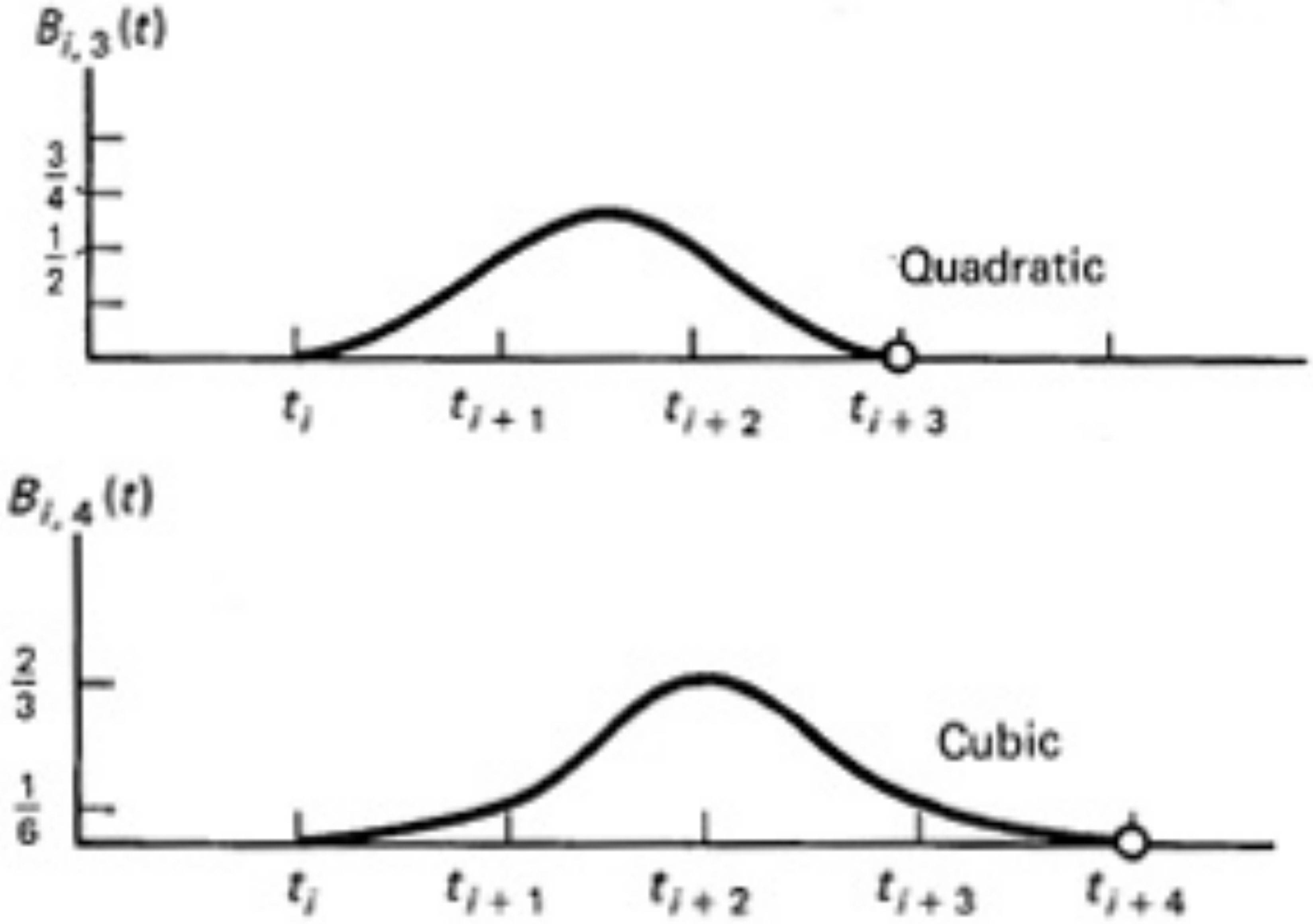
$$B_{i,1}(t) = \begin{cases} 1, & \mathrm{t_i \le t < t_{i+1}} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k-1} - t_i} B_{i+1,k-1}(t)$$

B-spline basis

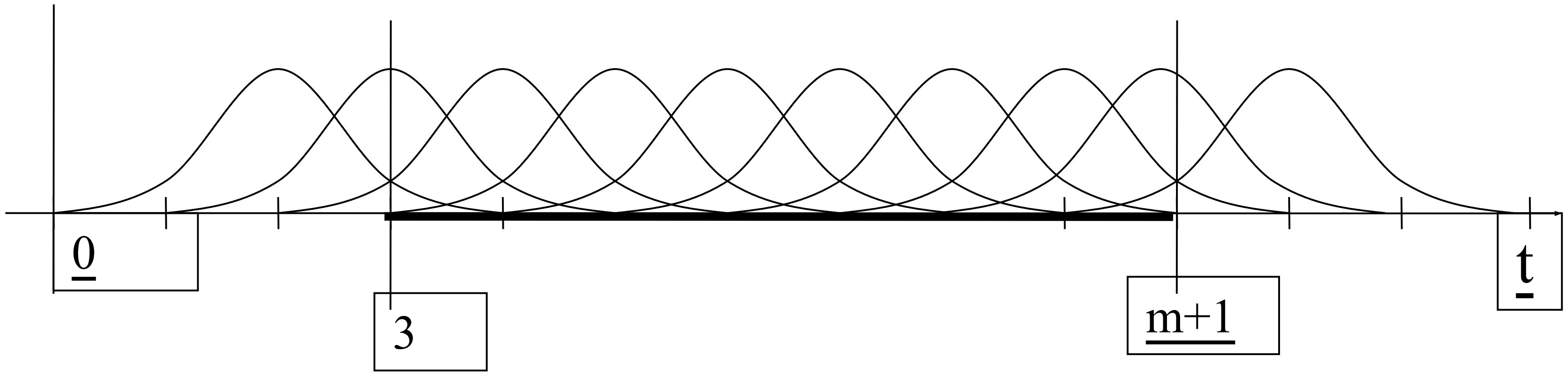


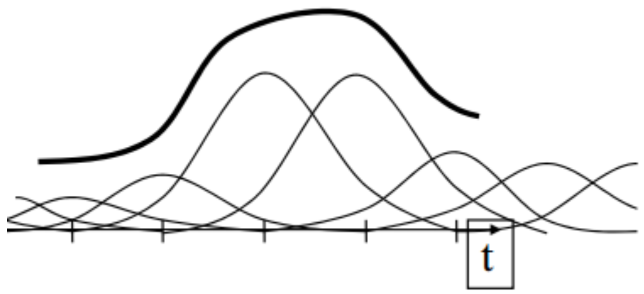
B-spline basis



Producing curves using B-splines

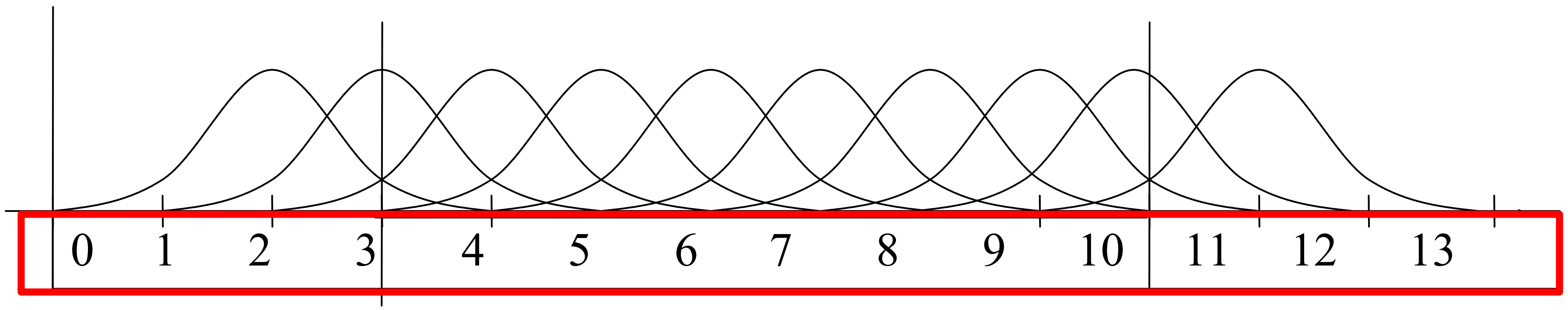
- The basis functions are multiplied by the control points to define arbitrary curves





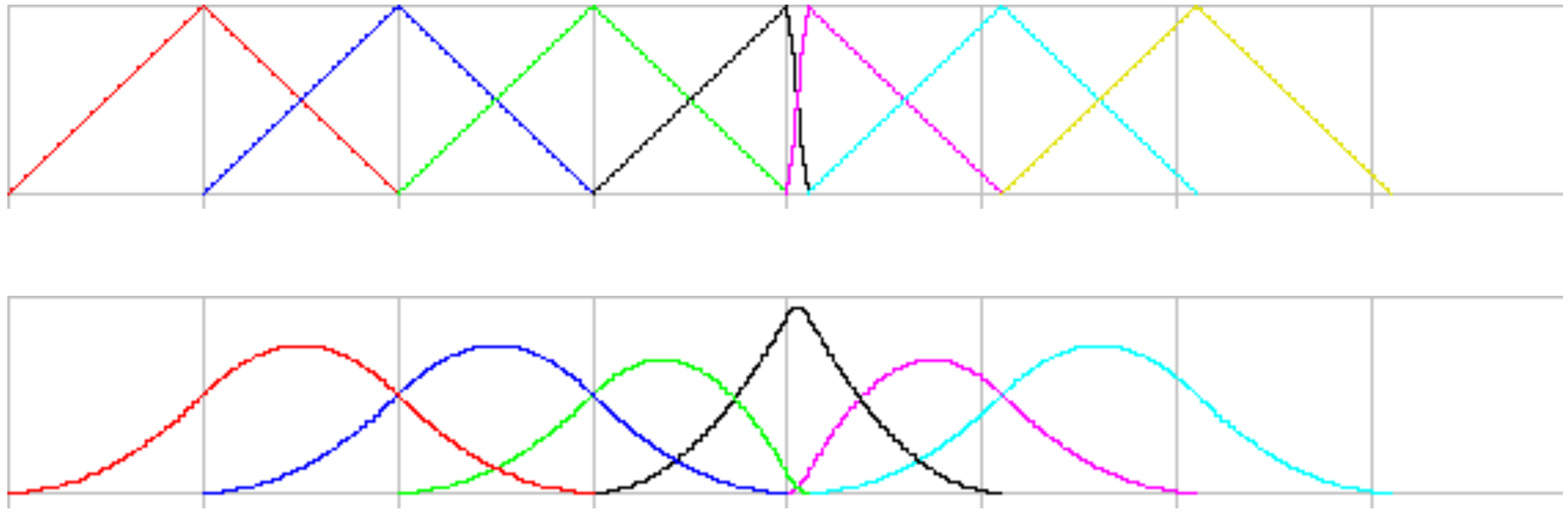
Knots

- The knots produce a vector that defines the domain of the curve
- The knots must be in increasing order
- Not necessarily uniform spacing
- If uniformly sampled and degree is 3 we have a uniform cubic B-spline



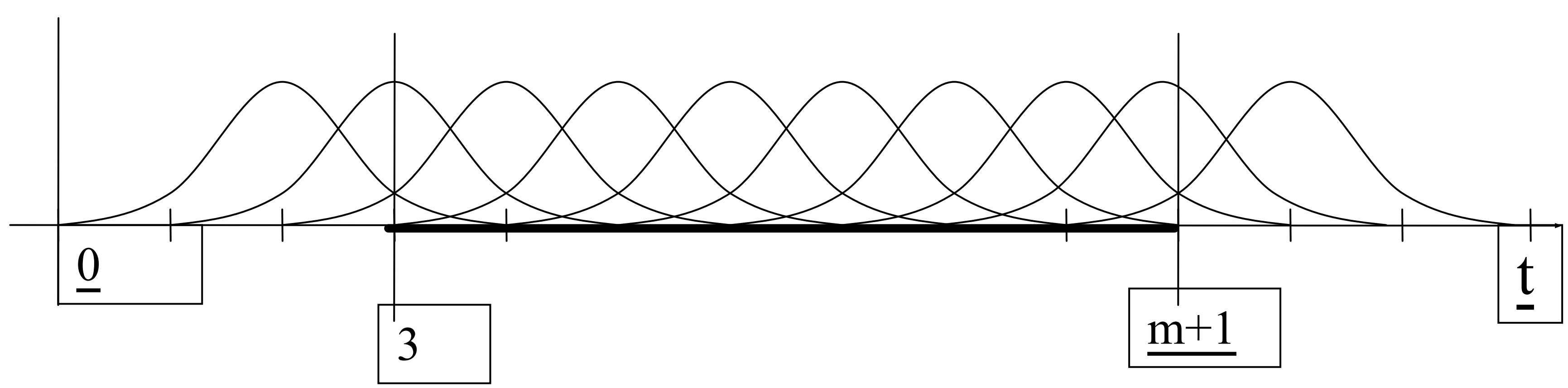
Knots

- Non-uniform knots:



B-spline terms

- Order k : the number of control points affecting the sampled value
- Degree $k-1$: the degree of the basis function polynomial
- Control points $P_i \quad i = (0, \ldots, m)$
- Knots $t_j \quad (j = 0, \ldots, n)$
- An important rule: $n - m = k$
- The domain of the curve is $t_{k-1} \leq t \leq t_{m+1}$
- Below, k=4, m=9, domain is $t_3 \leq t \leq t_{10}$

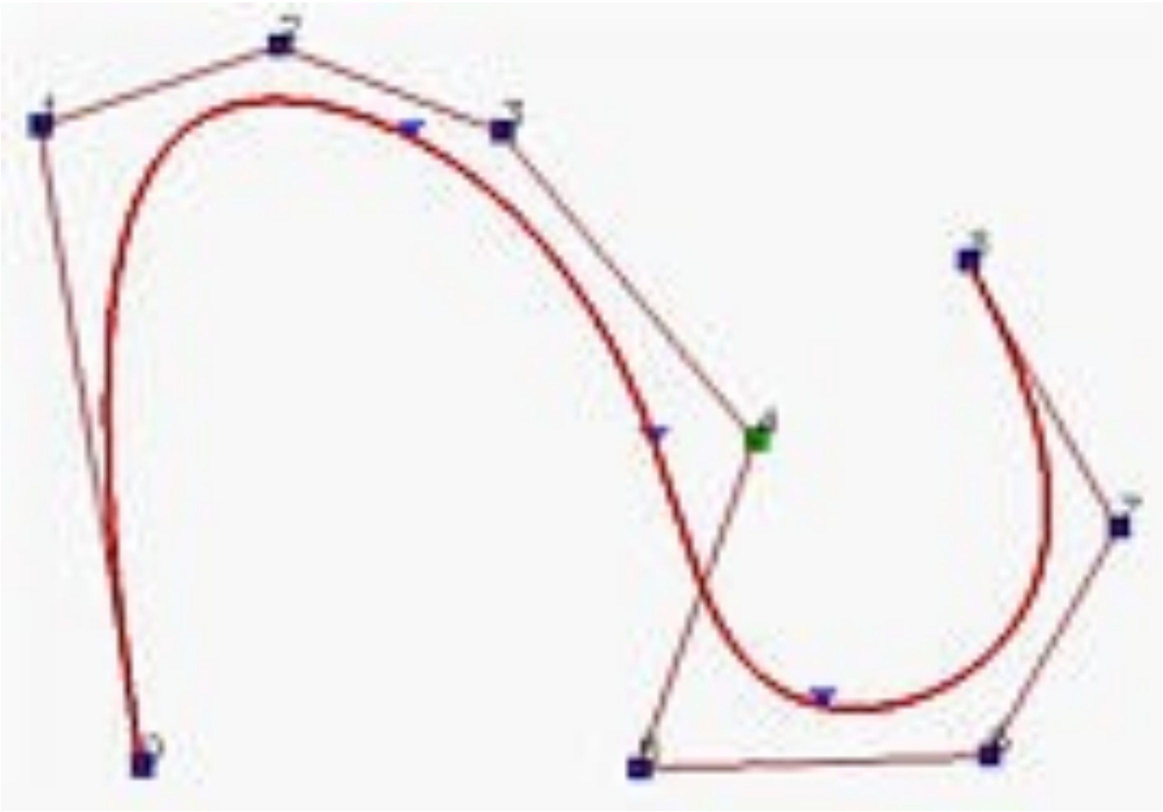


Clamped B-splines

- The first and last knot values are repeated with multiplicity equal to the order (degree + 1)

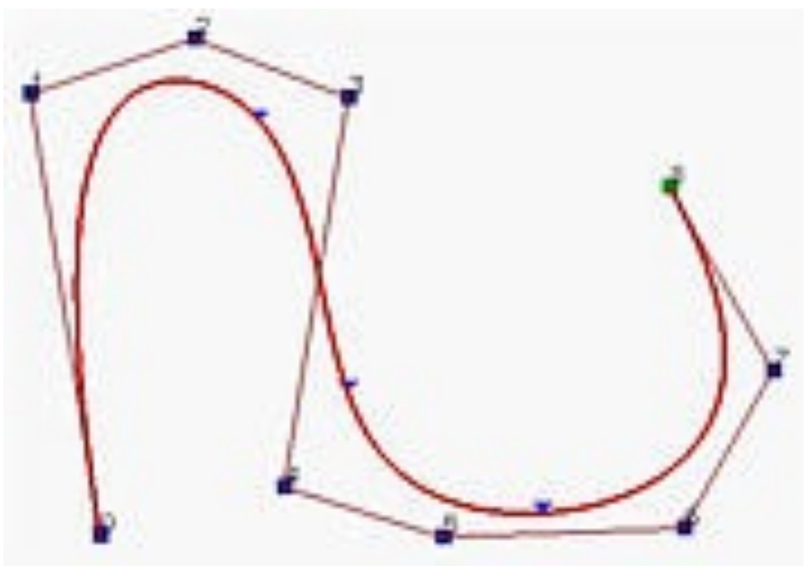
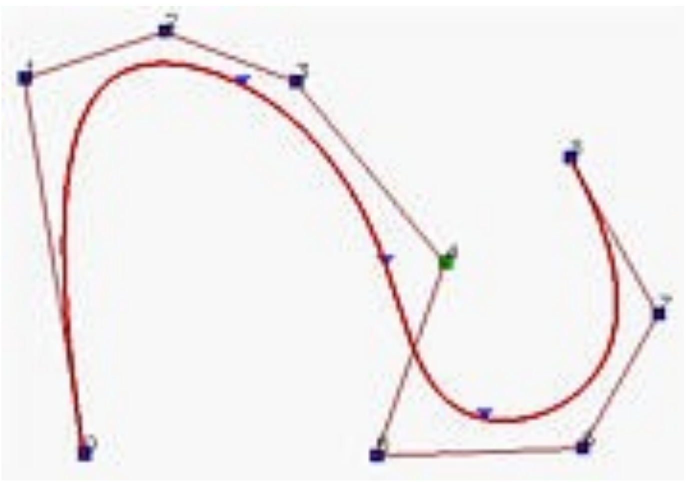
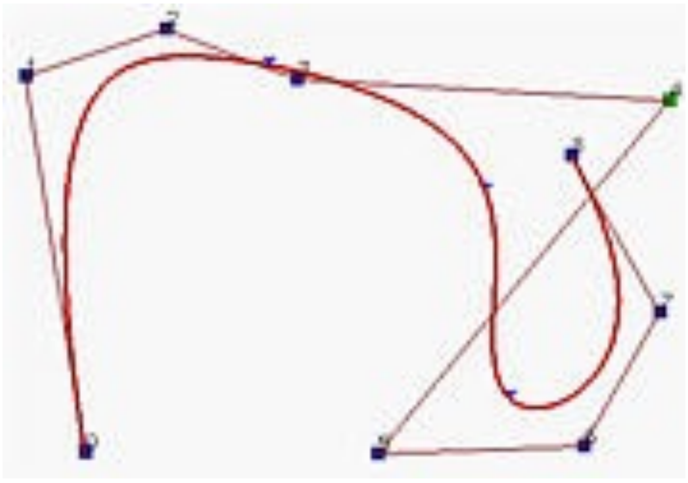
- The end points pass the control point

- For cubic bsplines, the multiplicity of the first / last knots must be 4 (repeated four times)



Controlling the shape of B-splines

- Moving the control points is the most obvious way to control bspline curves
- Changing the position of control point **P**i only affects the interval [ti, $ti+k$), where k is the order of a B-spline curve
  - Editing the shape through the knot vector is not very intuitive

Overview

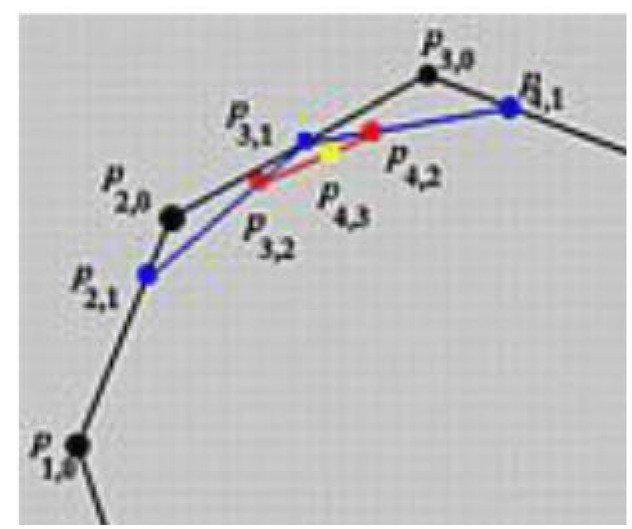
- More on Bézier and B-splines
  - de Casteljau's algorithm
  - General form of B-splines
  - **de Boor's algorithm**
  - **Knot insertion**
- NURBS
- Subdivision surfaces

de Boor's algorithm

- B-spline version of de Casteljau's algorithm
- A precise method to evaluate the curve
- Starting from control points and parameter value t, recursively solve:

$$\mathbf{P}_i^r = (1 - a_{i,r})\mathbf{P}_{i-1}^{r-1} + a_{i,r}\mathbf{P}_i^{r-1}$$

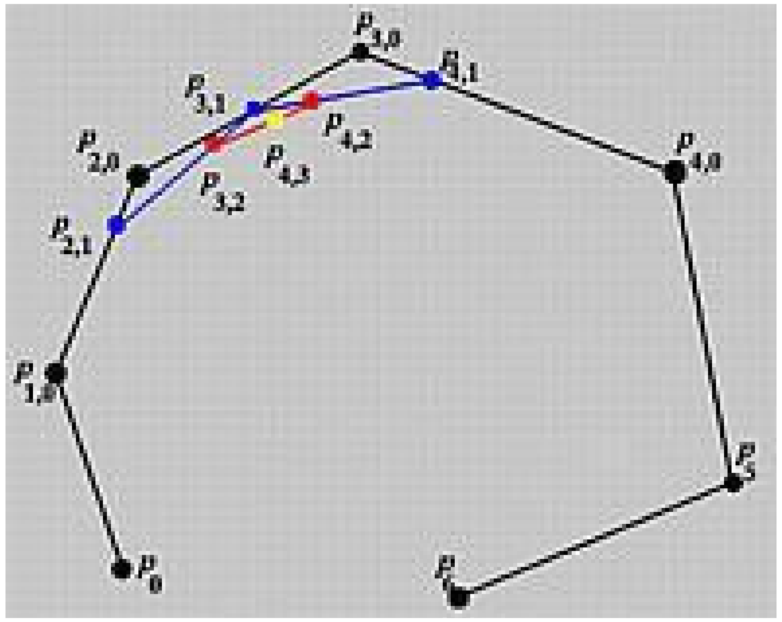
$$a_{i,r} = \frac{t - t_i}{t_{i+k-1-r} - t_i}$$



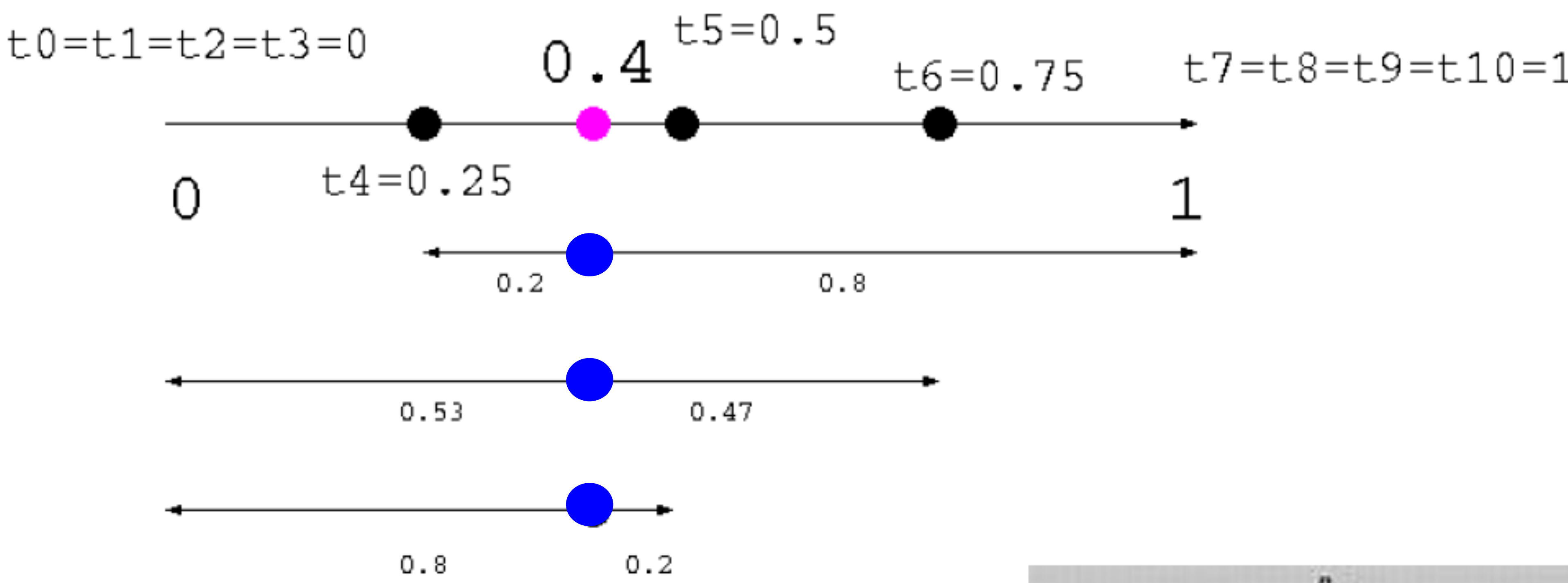
Example

- Assume we have a cubic B-spline with knot vector:
  $[0, 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1, 1]$
- Computing the point at $t = 0.4$

- Then $t_4 \leq t \leq t_5$ and the control points that affect the final position are P_4, P_3, P_2, P_1



Example



$$a_{4,1} = (t - t_4)/(t_{4+3} - t_4) = 0.2$$

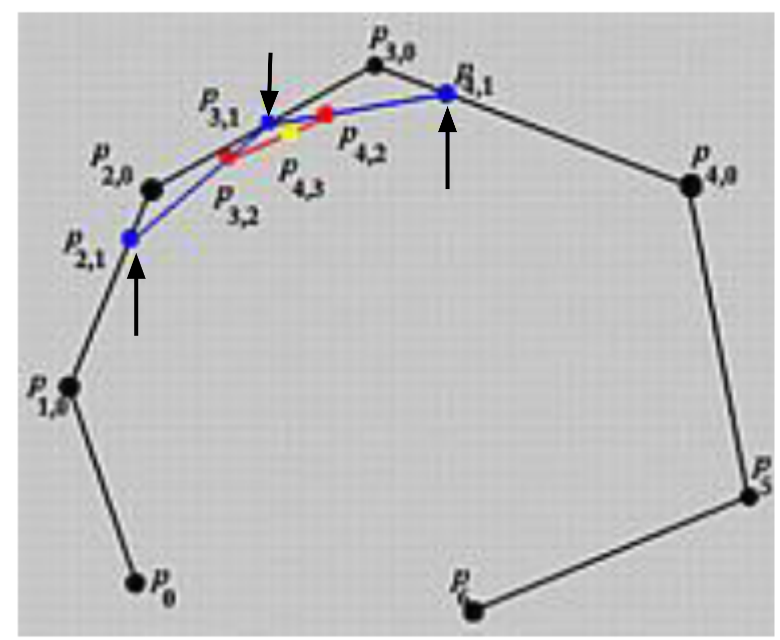
$$a_{3,1} = (t - t_3)/(t_{3+3} - t_3) = 0.53$$

$$a_{2,1} = (t - t_2)/(t_{2+3} - t_2) = 0.8$$

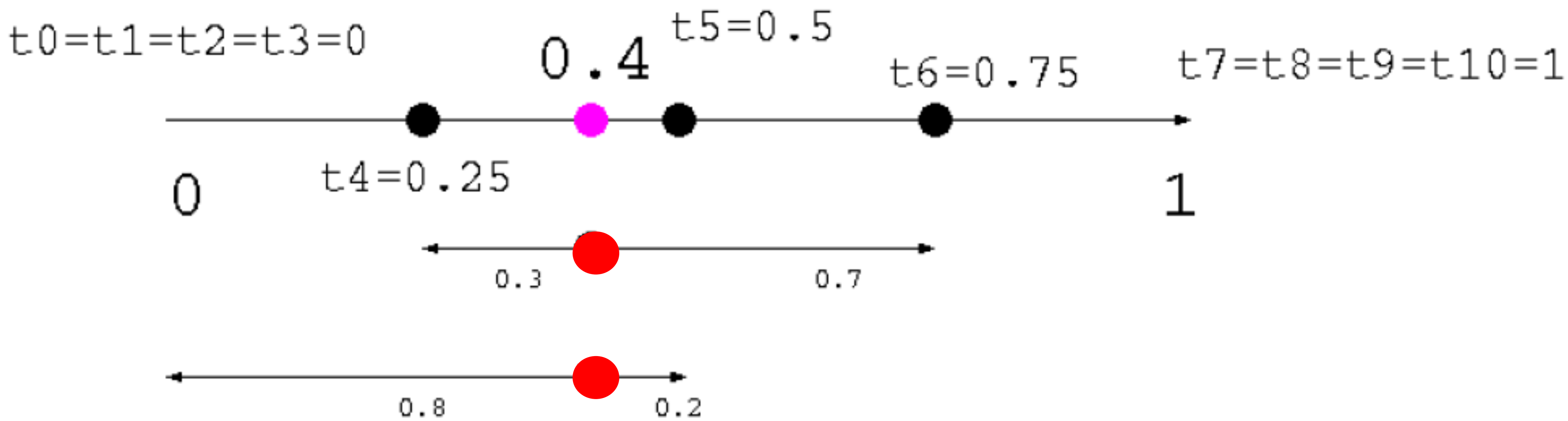
$$\mathbf{P}_{4,1} = (1 - a_{4,1})\mathbf{P}_{3,0} + a_{4,1}\mathbf{P}_{4,0}$$

$$\mathbf{P}_{3,1} = (1 - a_{3,1})\mathbf{P}_{2,0} + a_{3,1}\mathbf{P}_{3,0}$$

$$\mathbf{P}_{2,1} = (1 - a_{4,1})\mathbf{P}_{1,0} + a_{2,1}\mathbf{P}_{2,0}$$



Example

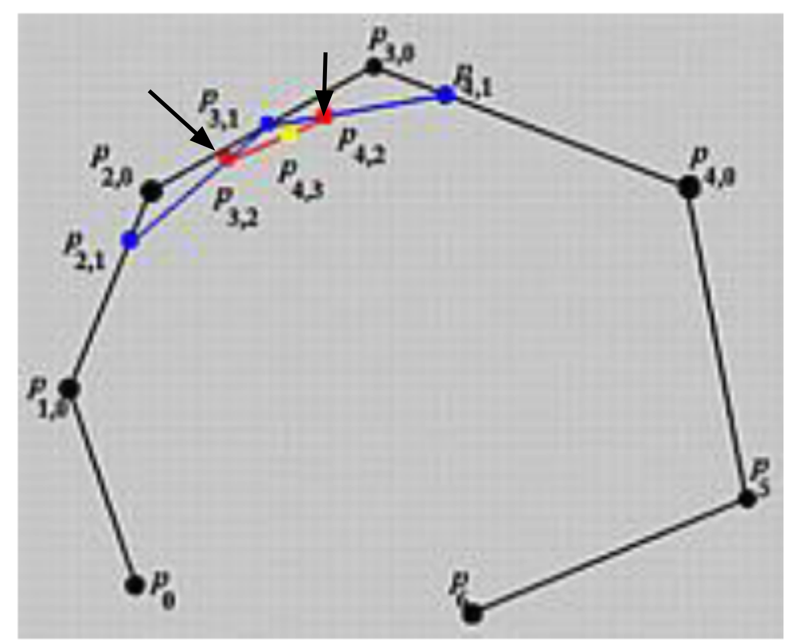


$$a_{4,2} = (t - t_4) / (t_{4+3-1} - t_4) = 0.3$$

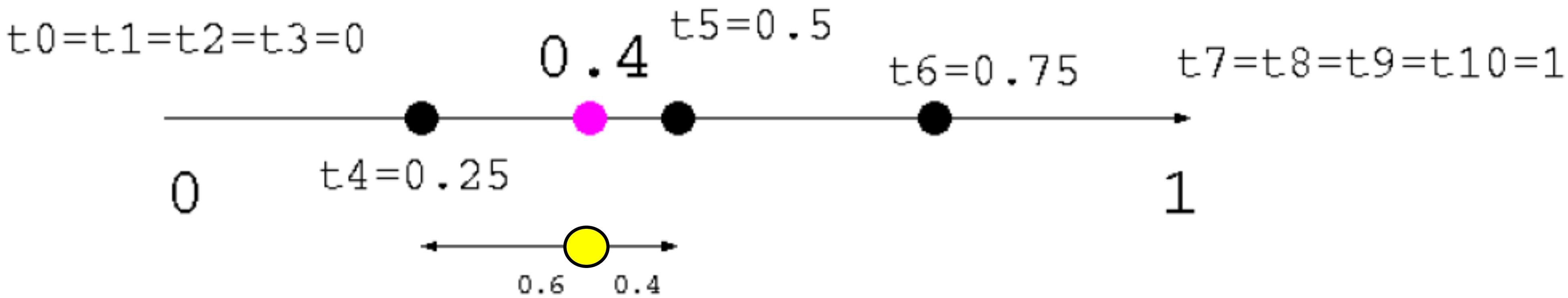
$$a_{3,2} = (t - t_3) / (t_{3+3-1} - t_3) = 0.8$$

$$\mathbf{P}_{4,2} = (1 - a_{4,2})\mathbf{P}_{3,1} + a_{4,2}\mathbf{P}_{4,1}$$

$$\mathbf{P}_{3,2} = (1 - a_{3,2})\mathbf{P}_{2,1} + a_{3,2}\mathbf{P}_{3,1}$$



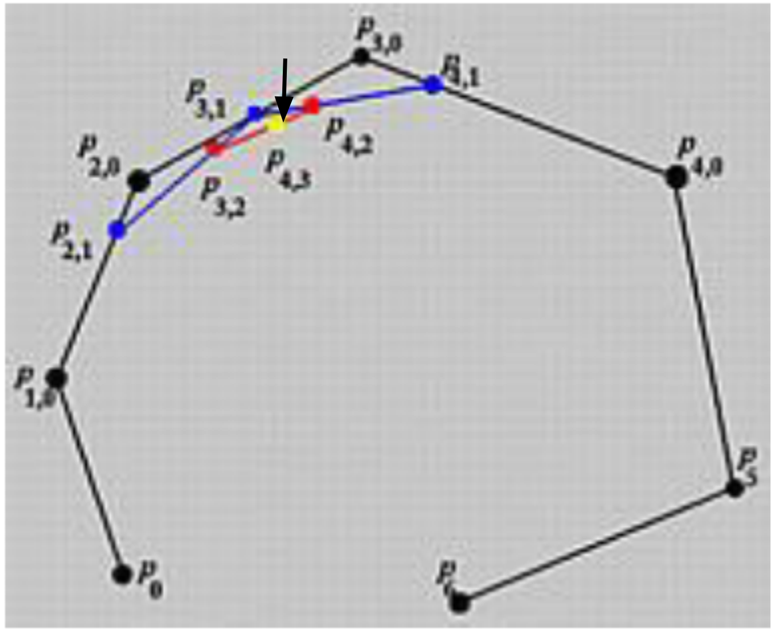
Example



$$a_{4,3} = (u - u_4)/(u_{4+3-2} - u_4) = 0.6$$

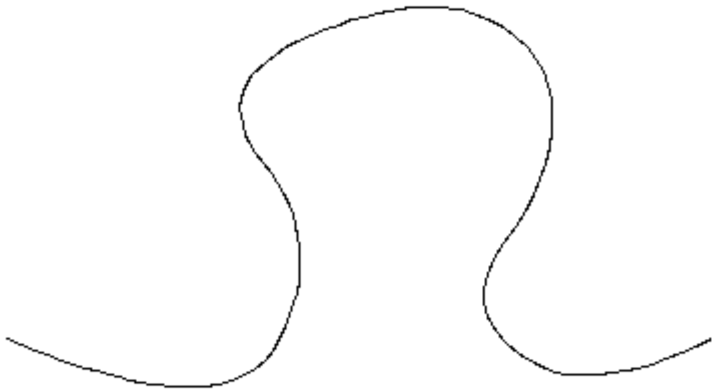
$$\mathbf{P}_{4,3} = (1 - a_{4,3})\mathbf{P}_{3,2} + a_{4,2}\mathbf{P}_{4,2}$$

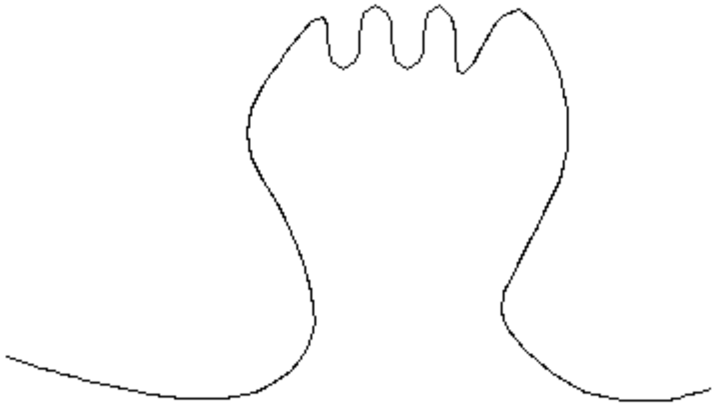
$$= 0.4\mathbf{P}_{3,2} + 0.6\mathbf{P}_{4,2}$$



What if you want to edit some details?

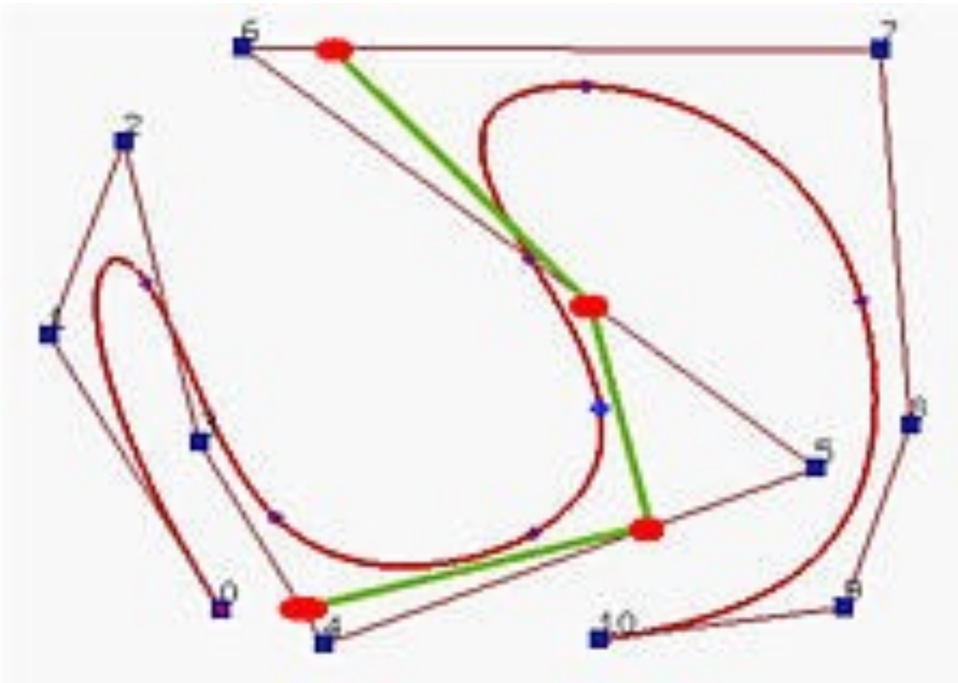
- You might want to add some high resolution details in a particular area whilst leaving the rest of the curve unchanged





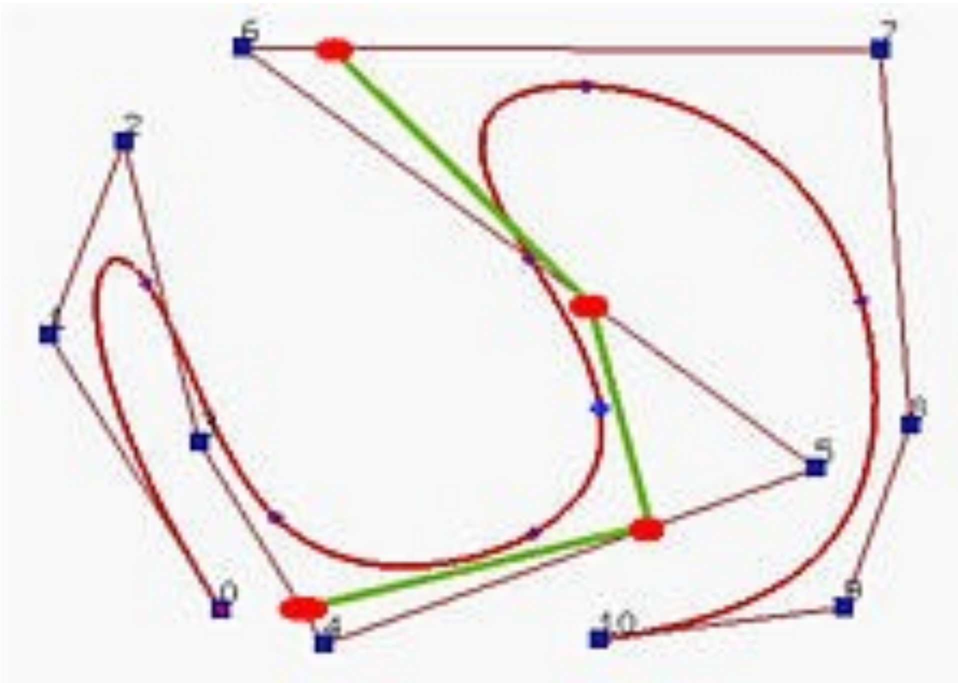
Knot insertion

- We can do this by knot Insertion
- New knots can be added without changing the shape of the curve
- Because of the basic rule n-m = k (n+1: number of knots, m +1: the number control points, k: order) the number of control points will also increase



Knot insertion

- For a curve of degree f we remove f-1 points and add f points
- i.e. for a cubic B-spline, remove 2 points and add 3 points



Knot insertion

- If the new knot t is inserted into the span $[t_j, t_{j+1})$,

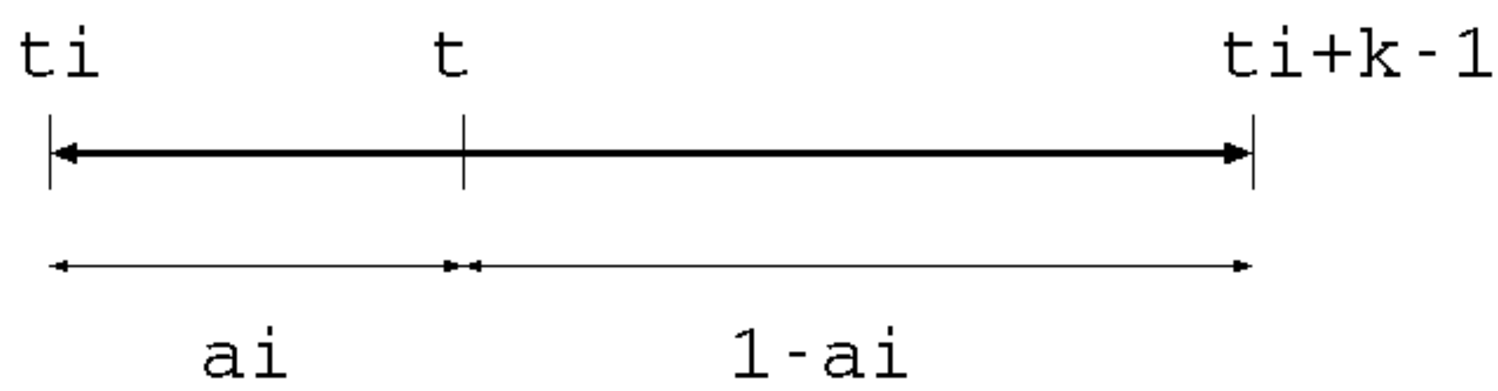
the new control points can be computed by

$$\mathbf{Q_i} = (1 - a_i)\mathbf{P_{i-1}} + a_i\mathbf{P_i}$$

where Qi is the new control point and a_i is computed by

$$a_i = \frac{t - t_i}{t_{i+k-1} - t_i} \quad \text{for } j\text{-}k + 2 \le i \le j$$

P*j-k+1*, P*j-k+2*, ..., P*j*-1, P*j* is replaced with P*j-k+1*, Q*j-k*+2, ..., Q*j*-1, Q*j* ,P*j*.

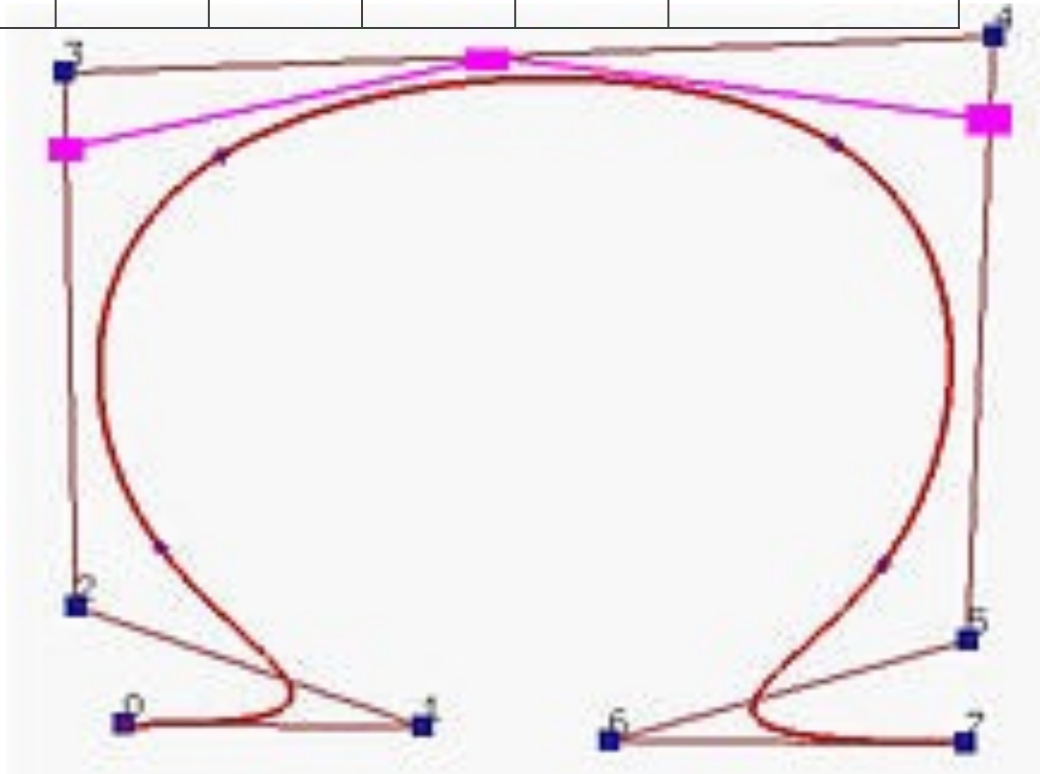


Example

- A bspline curve of degree 3 (k=4) having the following knots
- t=0.5 inserted

t_0 to t_3	t_4	t_5	t_6	t_7	t_8 to t_{11}
0	0.2	0.4	0.6	0.8	1

t_0 to t_3	t_4	t_5	**t_6**	t_7	t_8	t_9 to t_{12}
0	0.2	0.4	**0.5**	0.6	0.8	1

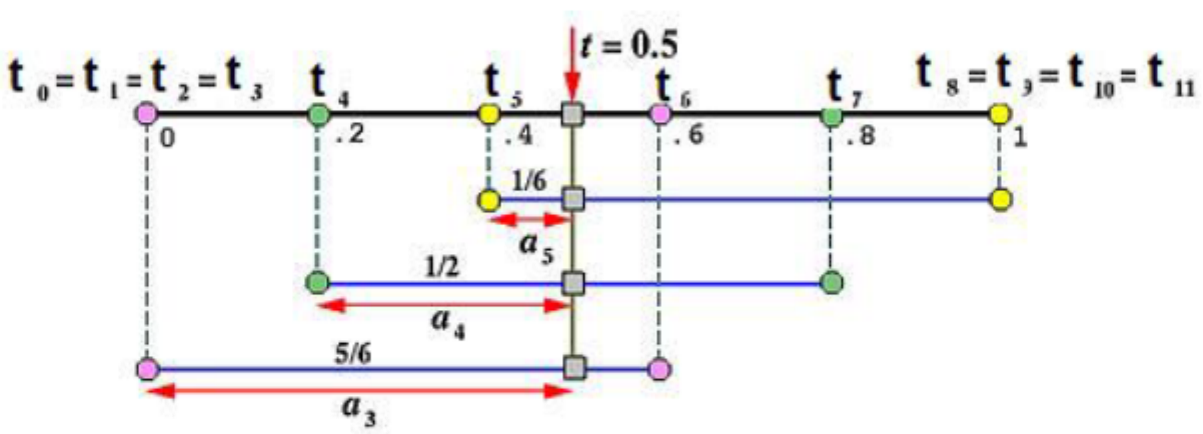


Example

- A bspline curve of degree 3 (k=4) having the following knots
- t=0.5 inserted

t_0 to t_3	t_4	t_5	t_6	t_7	t_8 to t_{11}
0	0.2	0.4	0.6	0.8	1

t_0 to t_3	t_4	t_5	t_6	t_7	t_8	t_9 to t_{12}
0	0.2	0.4	**0.5**	0.6	0.8	1



$$a_5 = \frac{t - t_5}{t_8 - t_5} = \frac{0.5 - 0.4}{1 - 0.4} = \frac{1}{6}$$

$$a_4 = \frac{t - t_4}{t_7 - t_4} = \frac{0.5 - 0.2}{0.8 - 0.2} = \frac{1}{2}$$

$$a_3 = \frac{t - t_3}{t_6 - t_3} = \frac{0.5 - 0}{0.6 - 0} = \frac{5}{6}$$

Example

- A bspline curve of degree 3 (k=4) having the following knots
- t=0.5 inserted

t_0 to t_3	t_4	t_5	t_6	t_7	t_8 to t_{11}
0	0.2	0.4	0.6	0.8	1

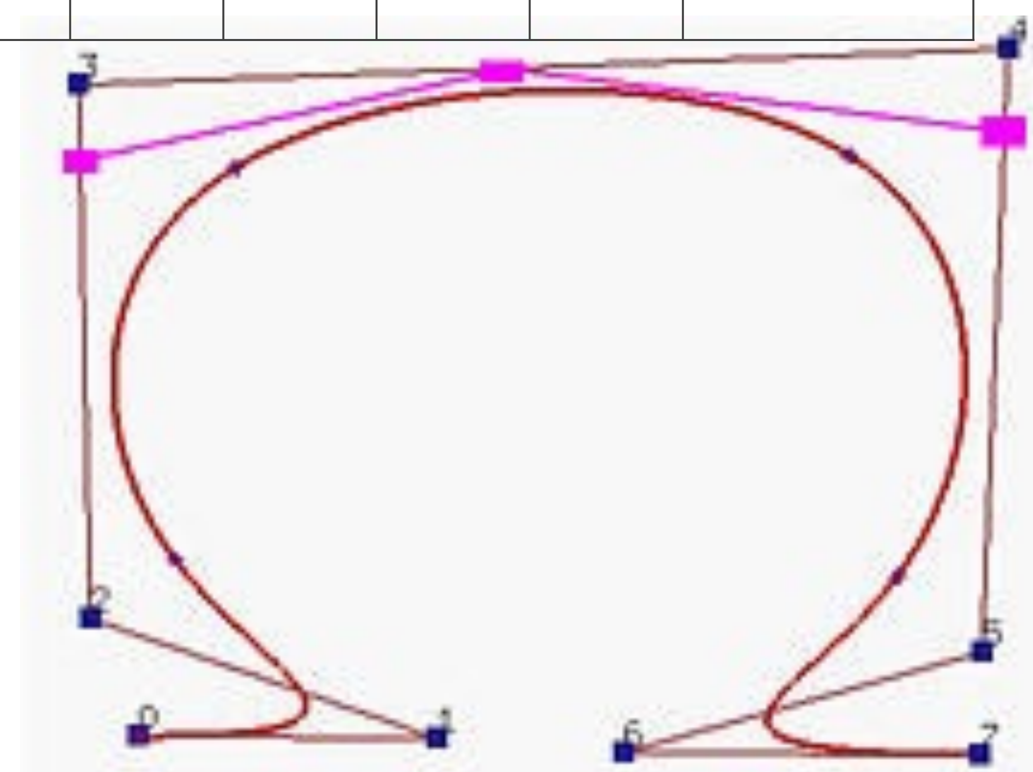
t_0 to t_3	t_4	t_5	t_6	t_7	t_8	t_9 to t_{12}
0	0.2	0.4	**0.5**	0.6	0.8	1

$$a_5 = \frac{t - t_5}{t_8 - t_5} = \frac{0.5 - 0.4}{1 - 0.4} = \frac{1}{6}$$

$$a_4 = \frac{t - t_4}{t_7 - t_4} = \frac{0.5 - 0.2}{0.8 - 0.2} = \frac{1}{2}$$

$$a_3 = \frac{t - t_3}{t_6 - t_3} = \frac{0.5 - 0}{0.6 - 0} = \frac{5}{6}$$

$$\begin{aligned} \mathbf{Q}_5 &= \left(1 - \frac{1}{6}\mathbf{P}_4\right) + \frac{1}{6}\mathbf{P}_5 \\ \mathbf{Q}_4 &= \left(1 - \frac{1}{2}\mathbf{P}_3\right) + \frac{1}{2}\mathbf{P}_4 \\ \mathbf{Q}_3 &= \left(1 - \frac{5}{6}\mathbf{P}_2\right) + \frac{5}{6}\mathbf{P}_3 \end{aligned}$$



http://i33www.ira.uka.de/applets/mocca/html/noplugin/curves.html

Summary of B-splines

- Knot vector defines the domain
- Evaluation by de Boor's algorithm
- Controlling the shape by the control points
- Clamping the points by increasing the multiplicity of the knots at the end points
- Increase the resolution by knot insertion

Overview

- More on Bézier and B-splines
  - de Casteljau's algorithm
  - General form of B-splines
  - de Boor's algorithm
  - Knot insertion
- **NURBS**
- Subdivision surfaces

NURBS (Non-uniform rational B-spline)

- Standard curves/surface representation in computer aided design

$$C(t) = \frac{\sum_{i=0}^{n} B_{i,k}(t) w_i \mathbf{P}_i}{\sum_{i=0}^{n} B_{i,k}(t) w_i}$$

P_i: control points

$B_{i,k}$: Bspline basis of order k

w_i : weights

Benefits of using NURBS

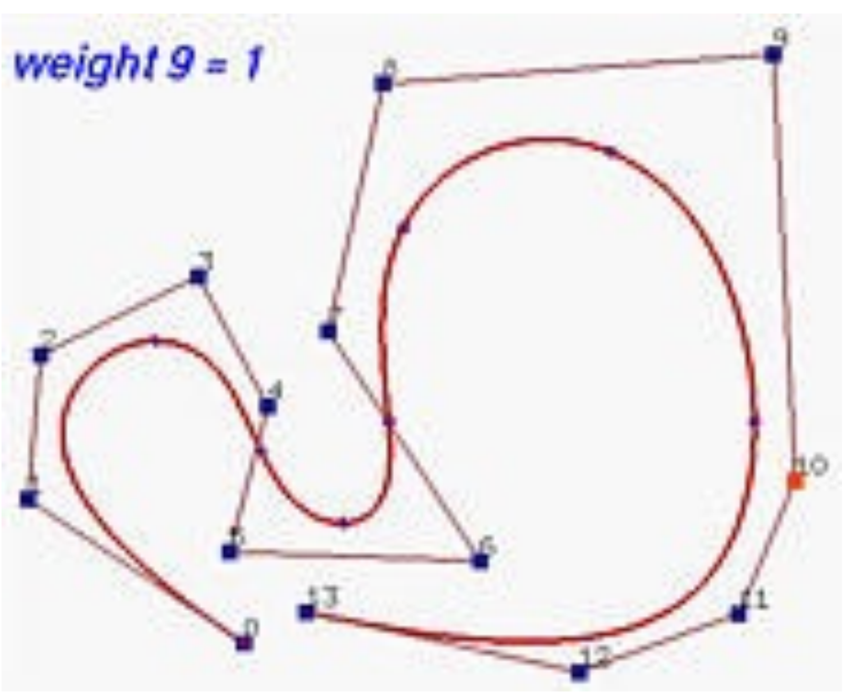
- More degrees of freedom to control the curve (can control the weights)

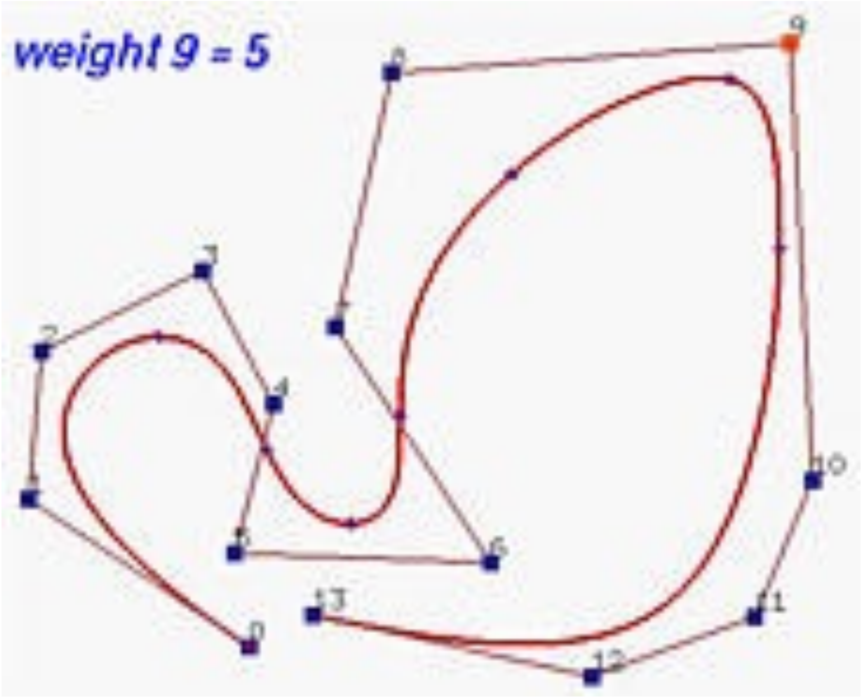
- Invariant under perspective transformation
  - Can project the control points onto the screen and interpolate on the screen
  - Don't need to apply the perspective transformation to all the points on the curve

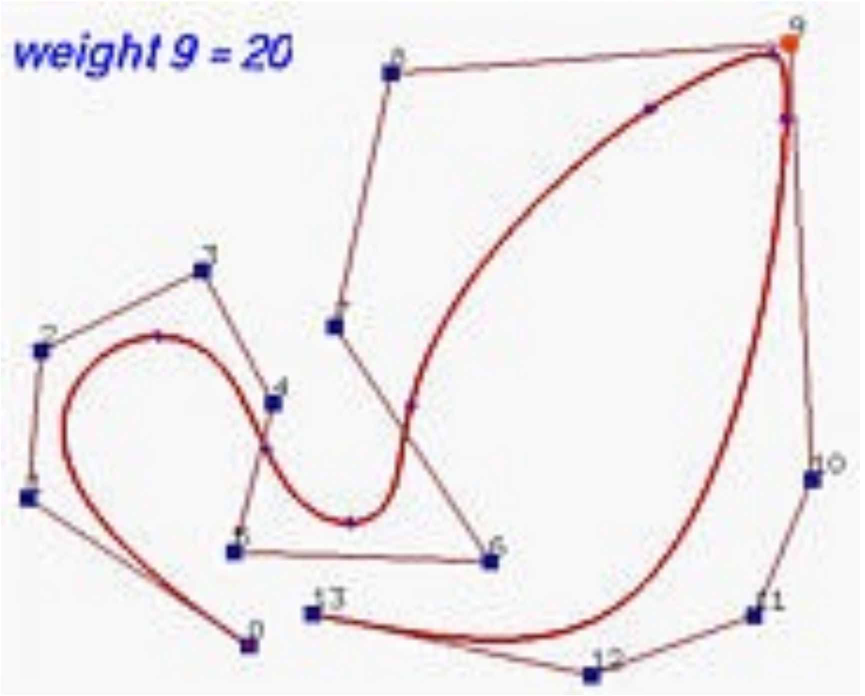
- Can model conic sections such as circles, ellipses and hyperbolas

Example of changing weights

- Increasing the weight will bring the curve closer to the corresponding control point





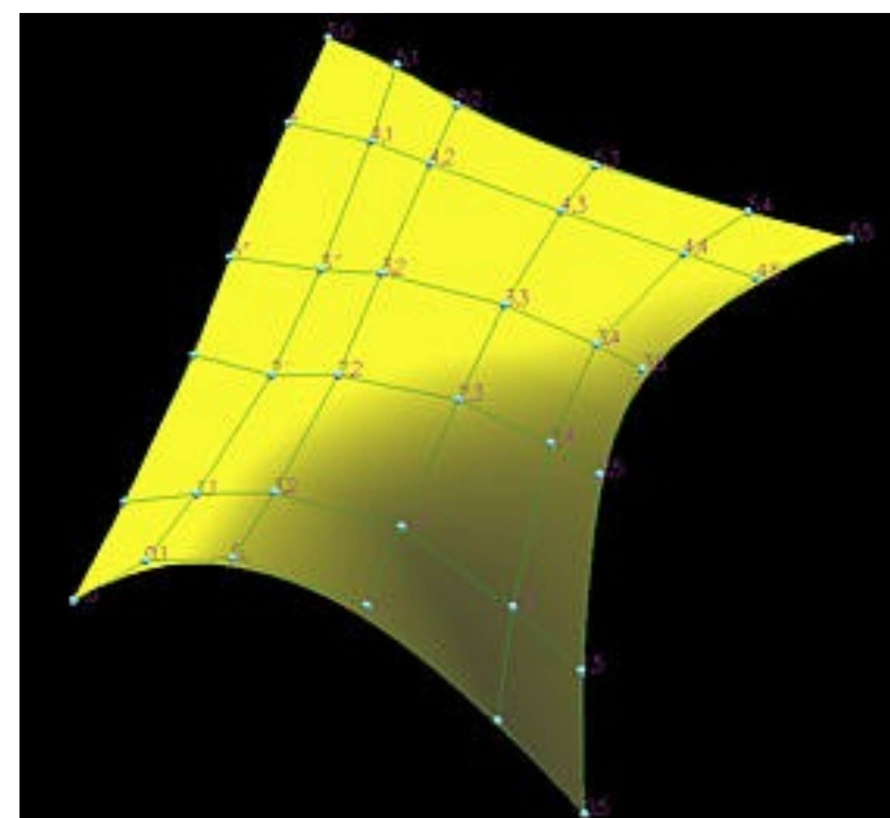


B-spline Surfaces

- Given the following information:
- a set of *m*+1 rows and *n*+1 control points $P_{i,j}$ where $0 \leq i \leq m$ and $0 \leq j \leq n$
- Corresponding knot vectors in the *u and v* direction,

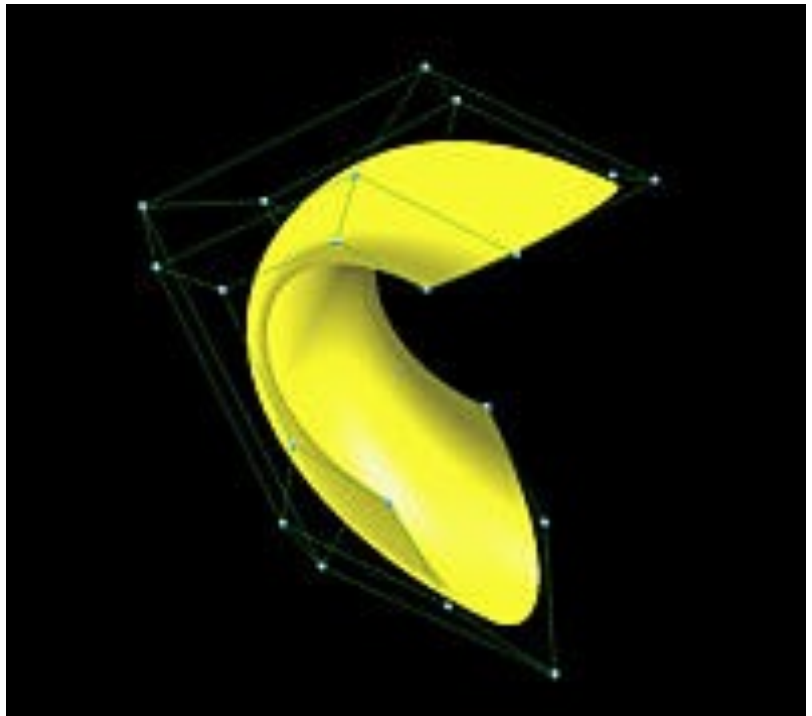
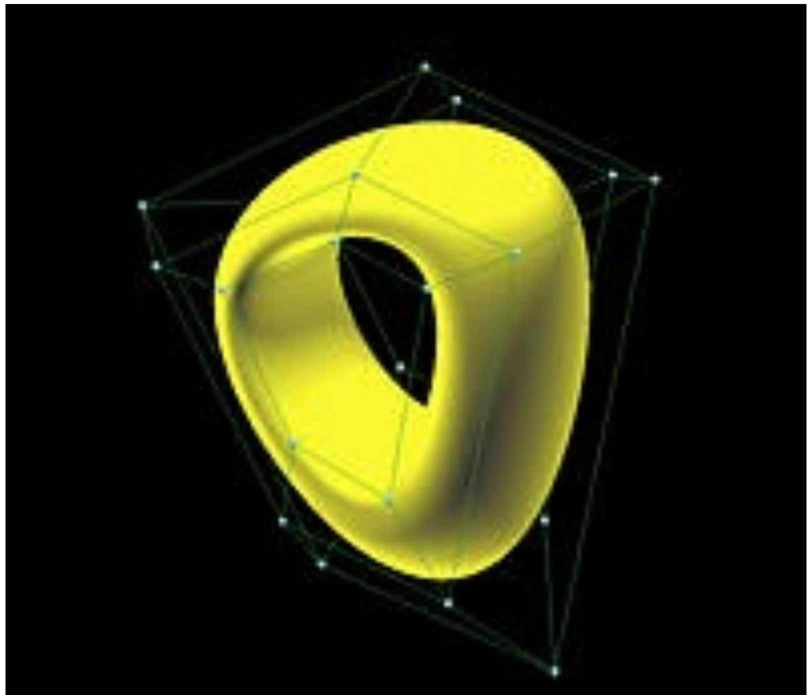
$$p(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{i,p}(u) B_{j,q}(v) \mathbf{P}_{i,j} \text{ : non – rational B - spline}$$

$$p(u,v) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{i,p}(u) B_{j,q}(v) \mathbf{P}_{i,j}}{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{i,p}(u) B_{j,q}(v)} \text{ : NURBS}$$



Clamped, Closed and Open B-spline Surfaces

- Since a B-spline curve can be clamped, closed or open, a B-spline surface can also have three types *in each direction*.
- That is, we could ask to have a B-spline surface clamped in the u-direction and closed in the v-direction.
- If a B-spline is clamped in both directions, then this surface passes though control points $\mathbf{p}_{0,0}$, $\mathbf{p}_{m,0}$, $\mathbf{p}_{0,n}$ and $\mathbf{p}_{m,n}$
- If a B-spline surface is closed in one direction, then the surface becomes a tube.
- Closed in two direction : torus
  - Problems handling objects of arbitrary topology, such as a ball, double torus


Overview

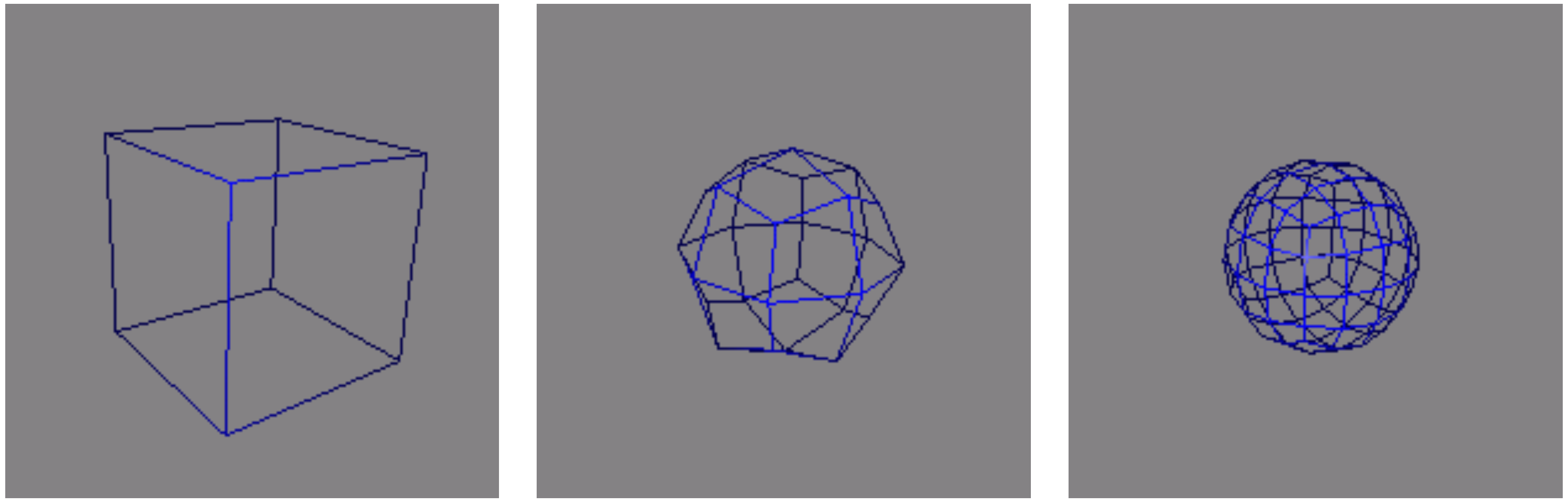
- More on Bézier and B-splines
  - de Casteljau's algorithm
  - General form of B-splines
  - de Boor's algorithm
  - Knot insertion
- NURBS
- **Subdivision surfaces**

Subdivision Surfaces

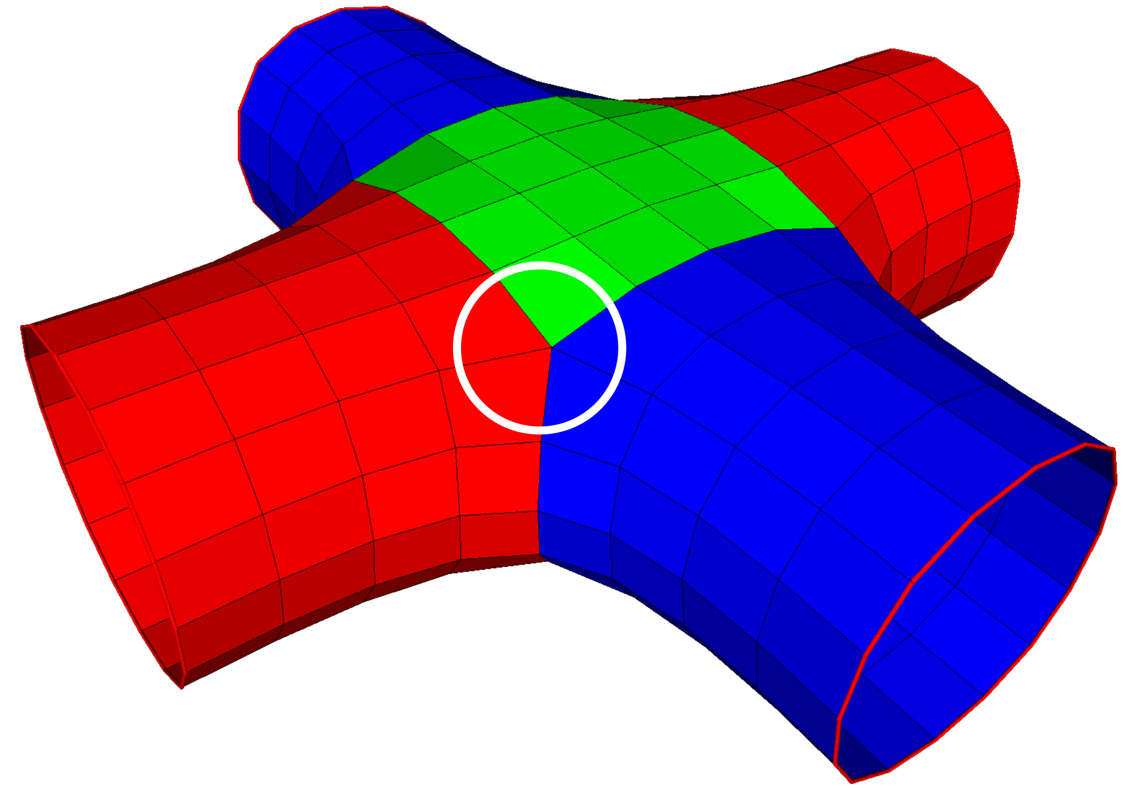
- A method to model smooth surfaces

3D subdivision surface

- Start with a rough shape first and subdivide it recursively
- Stop when the shape is smooth enough
- Used for modelling smooth surfaces



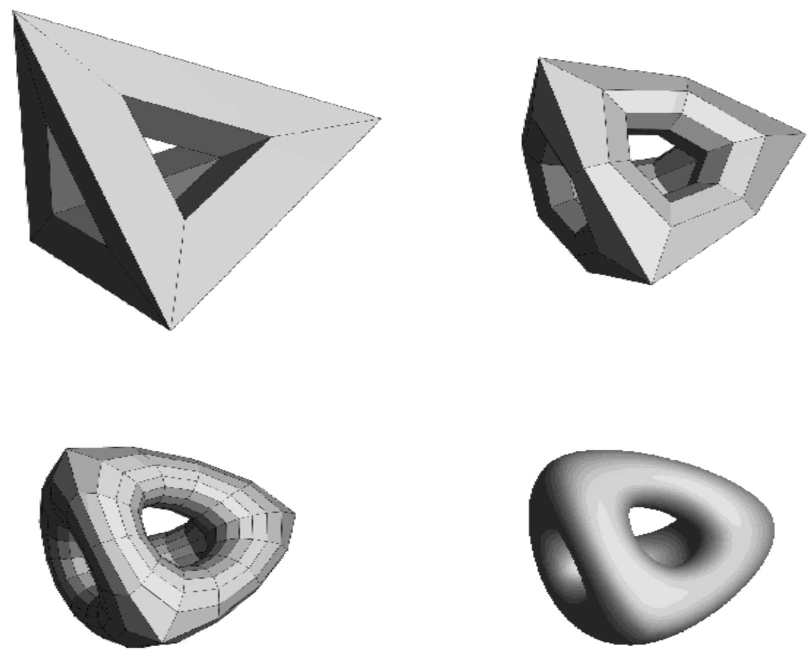
Motivation



- Shape modeling
  - Topological restrictions of NURBS surfaces
    - Plane, Cylinder, and Torus
    - It is difficult to maintain smoothness at seams of patchwork.
      - Example: hiding seams in Woody (*Toy Story*) [DeRose98]
  - NURBS also require the control nets consist of a regular rectangular grid of control points
- LOD in a scene
  - A coarse shape when far away, a smooth dense surface when closer to the camera

Subdivision surface

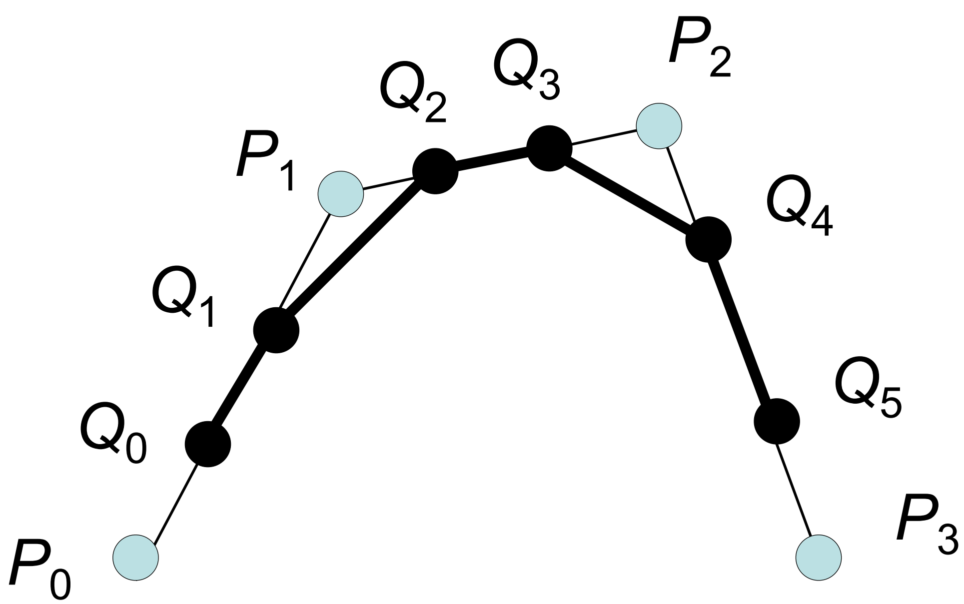
- Can handle arbitrary topology



Different Schemes

- Doo-Sabin ‘78
- Catmull-Clark ‘78
- Etc (Loop, Butterfly, and many others)

A Primer: Chaiken's Algorithm



$$Q_0 = \frac{3}{4}P_0 + \frac{1}{4}P_1$$

$$Q_1 = \frac{1}{4}P_0 + \frac{3}{4}P_1$$

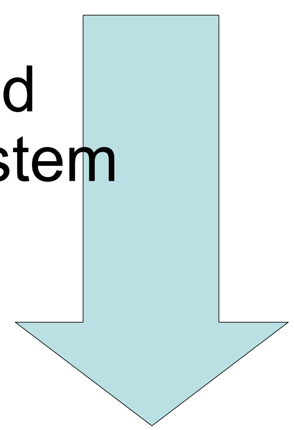
$$Q_2 = \frac{3}{4}P_1 + \frac{1}{4}P_2$$

$$Q_3 = \frac{1}{4}P_1 + \frac{3}{4}P_2$$

$$Q_4 = \frac{3}{4}P_2 + \frac{1}{4}P_3$$

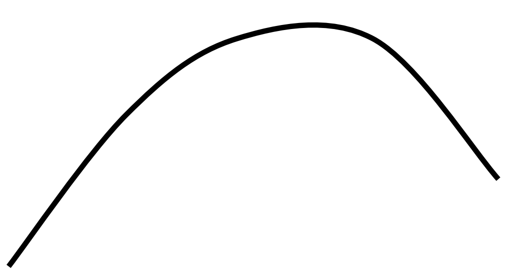
$$Q_5 = \frac{1}{4}P_2 + \frac{3}{4}P_3$$

Apply Iterated Function System



$$Q_{2i} = \frac{3}{4}P_i + \frac{1}{4}P_{i+1}$$

$$Q_{2i+1} = \frac{1}{4}P_i + \frac{3}{4}P_{i+1}$$

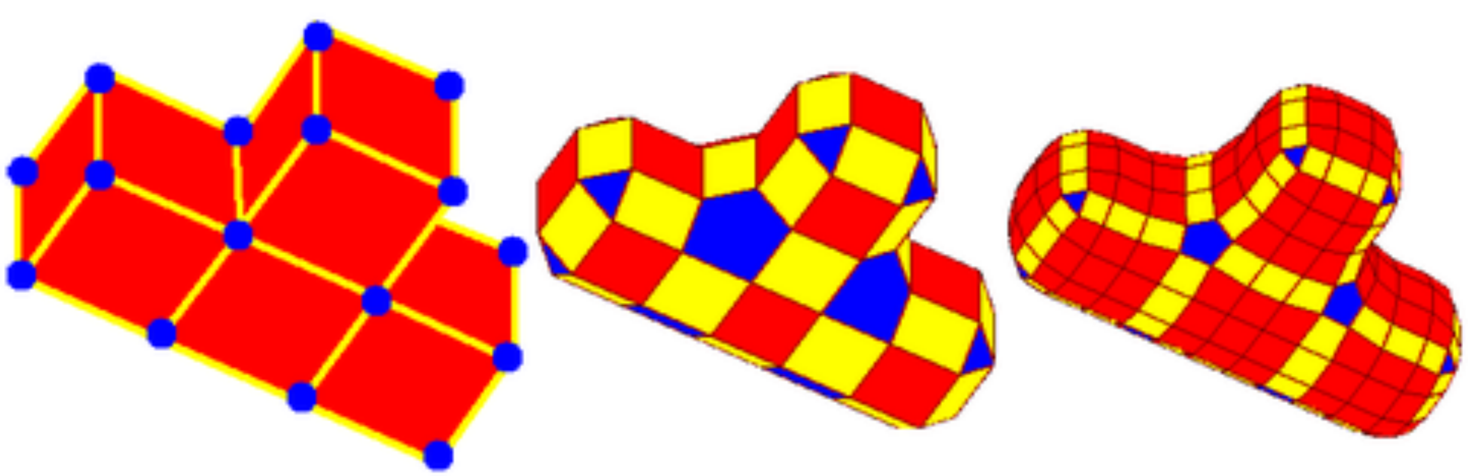


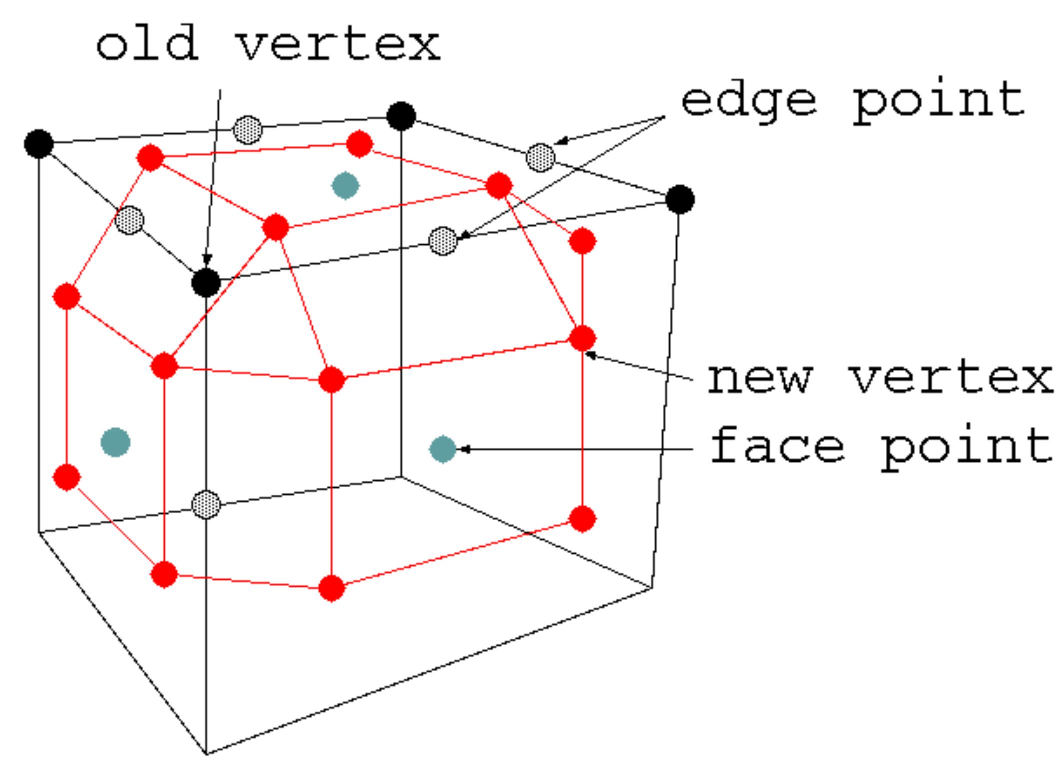
Limit Curve Surface

http://www.multires.caltech.edu/teaching/demos/java/chaikin.htm

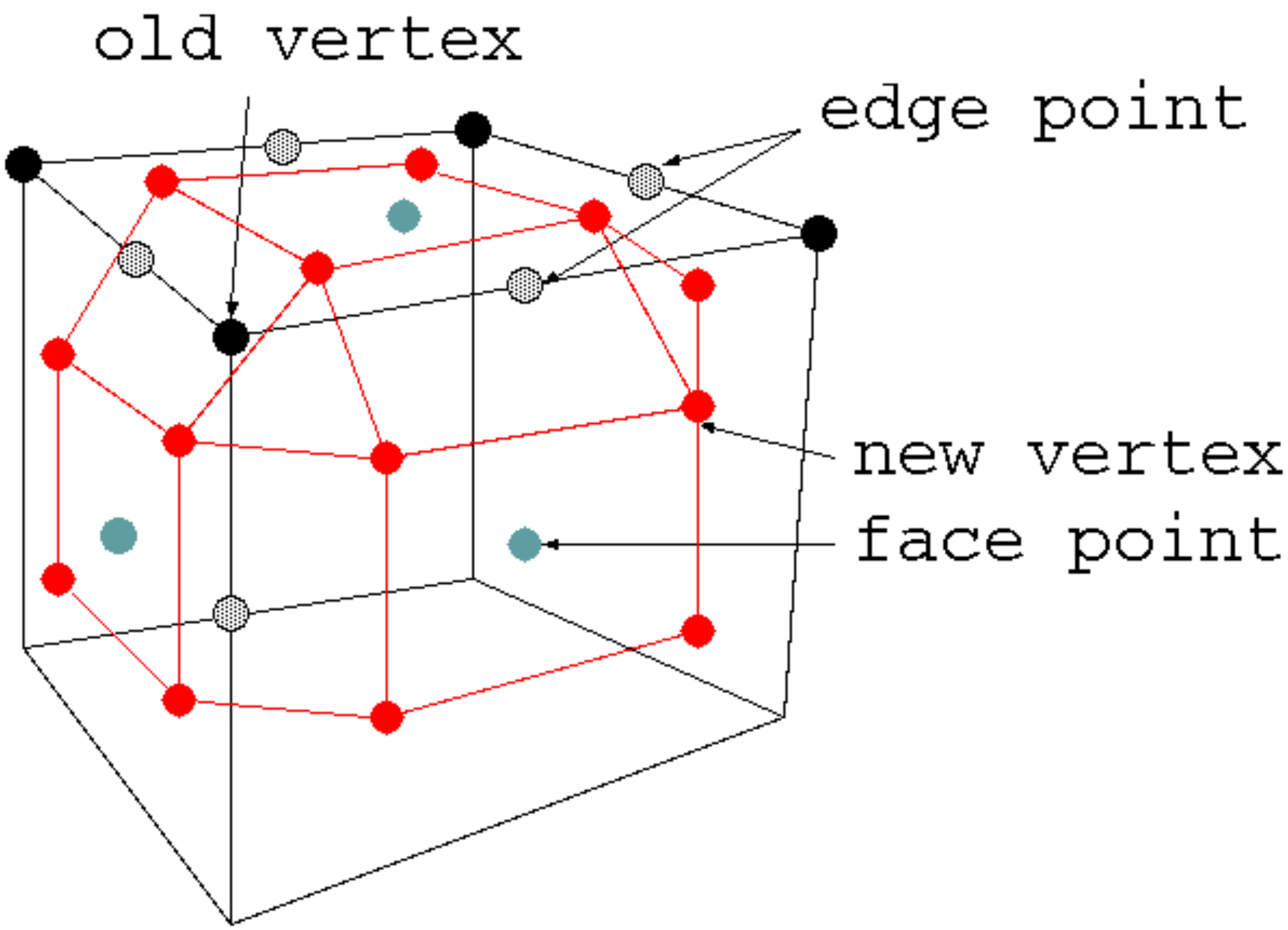
Doo-Sabin Subdivision

- An *edge point* is formed from the midpoint of each edge
- A *face point* is formed as the centroid of each polygon of the mesh.
- Finally, each vertex in the new mesh is formed as the average of
  - a vertex in the old mesh,
  - a face point for a polygon that touches that old vertex, and
  - the edge points for the two edges that belong to that polygon and touch that old vertex.





Doo-Sabin Subdivision

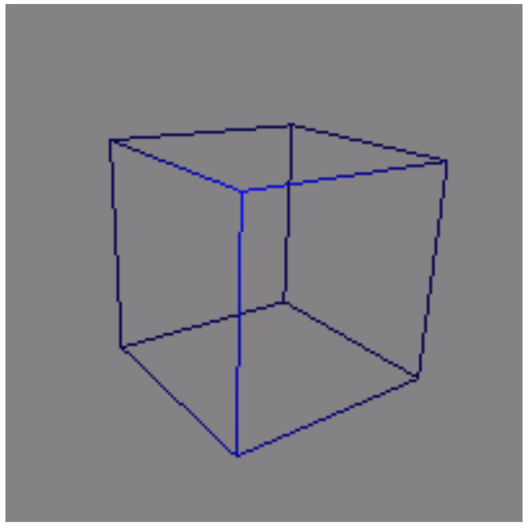
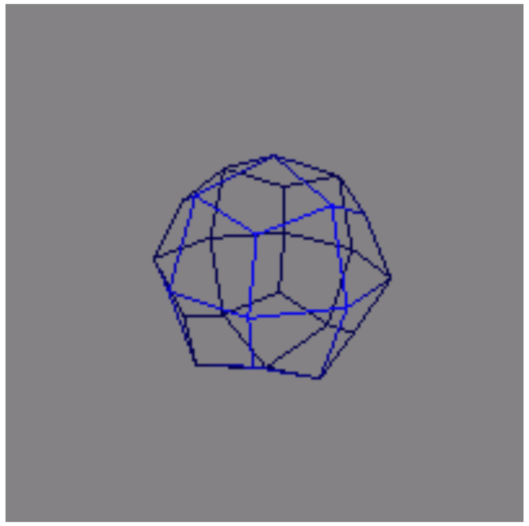
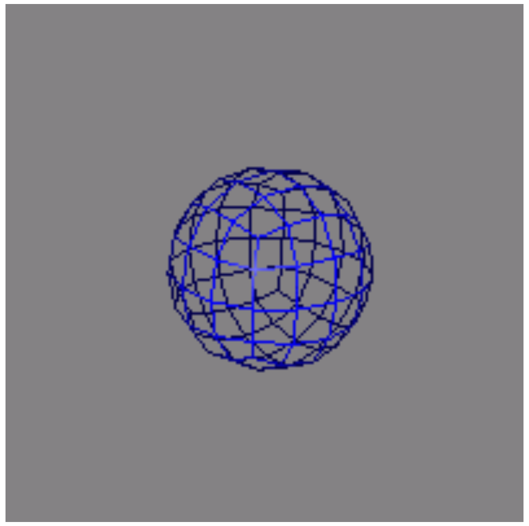


The new mesh, therefore, will

- create quadrilaterals for each edge in the old mesh,
- create a smaller n-sided polygon for each n-sided polygon in the old mesh, and
- create an n-sided polygon for each n-valence vertex (Valence being the number of edges that touch the vertex).

Catmull-Clark Subdivision

- A face with n edges are subdivided into n quadrilaterals
- Quads are better than triangles at capturing the symmetries of natural and man-made objects. Tube like surfaces (arms, legs, fingers) are easier to model.

Catmull-Clark Subdivision

FACE

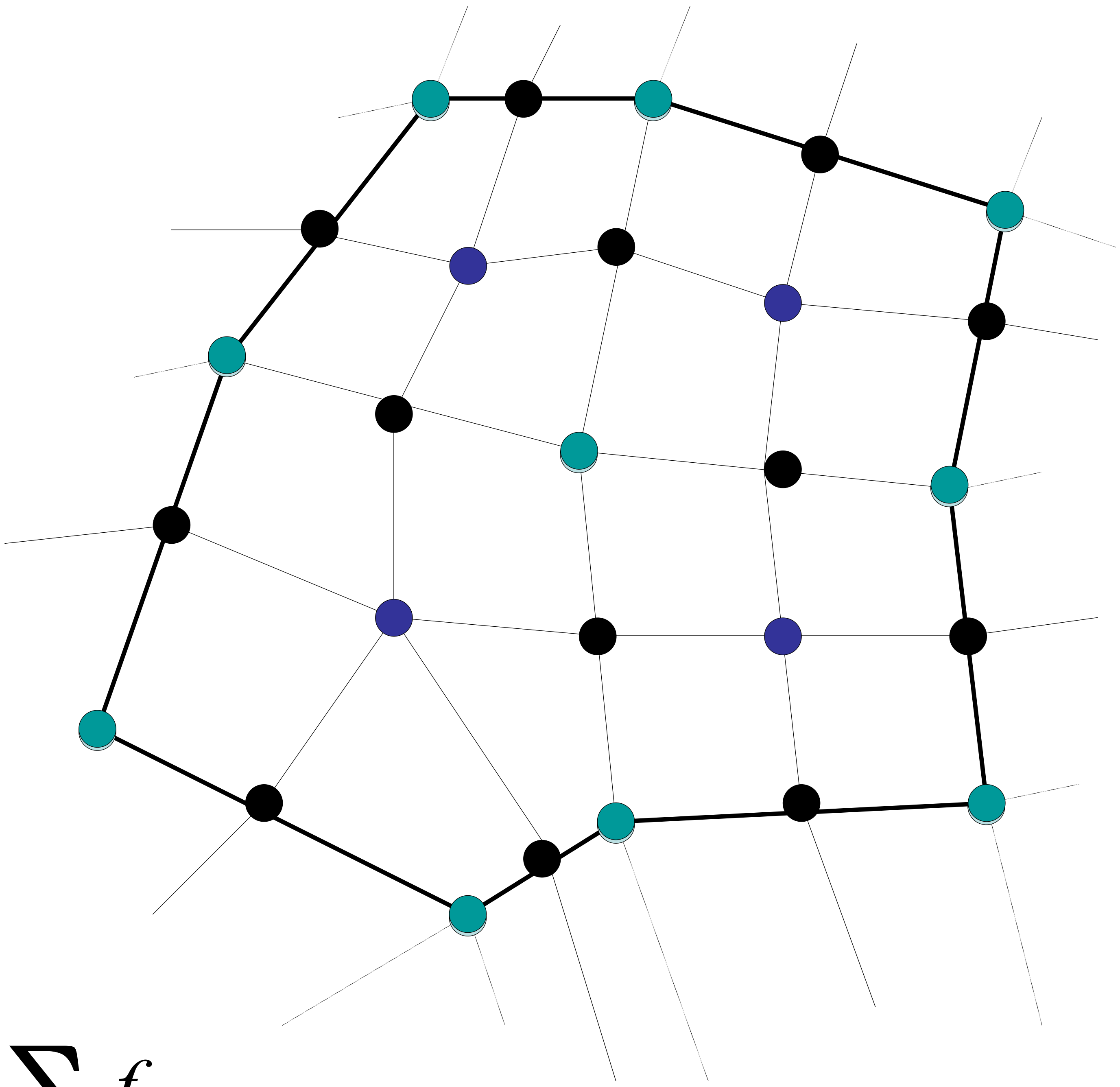
$$f = \frac{1}{n}\sum_{1}^{n} v_i$$

EDGE

$$e = \frac{v_1 + v_2 + f_1 + f_2}{4}$$

VERTEX

$$v_{i+1} = \frac{n-2}{n} v_i + \frac{1}{n^2}\sum_j e_j + \frac{1}{n^2}\sum_j f_j$$



Modelling with Catmull-Clark

- Subdivision produces smooth continuous surfaces.
- How can "sharpness" and creases be controlled in a modeling environment?

ANSWER: Define new subdivision rules for "creased" edges and vertices.

1. Tag Edges sharp edges.
2. If an edge is sharp, apply new sharp subdivision rules.
3. Otherwise subdivide with normal rules.

Sharp Edges…

- Tag Edges as "sharp" or "not-sharp"
  - n = 0 – "not sharp"
  - n > 0 – sharp

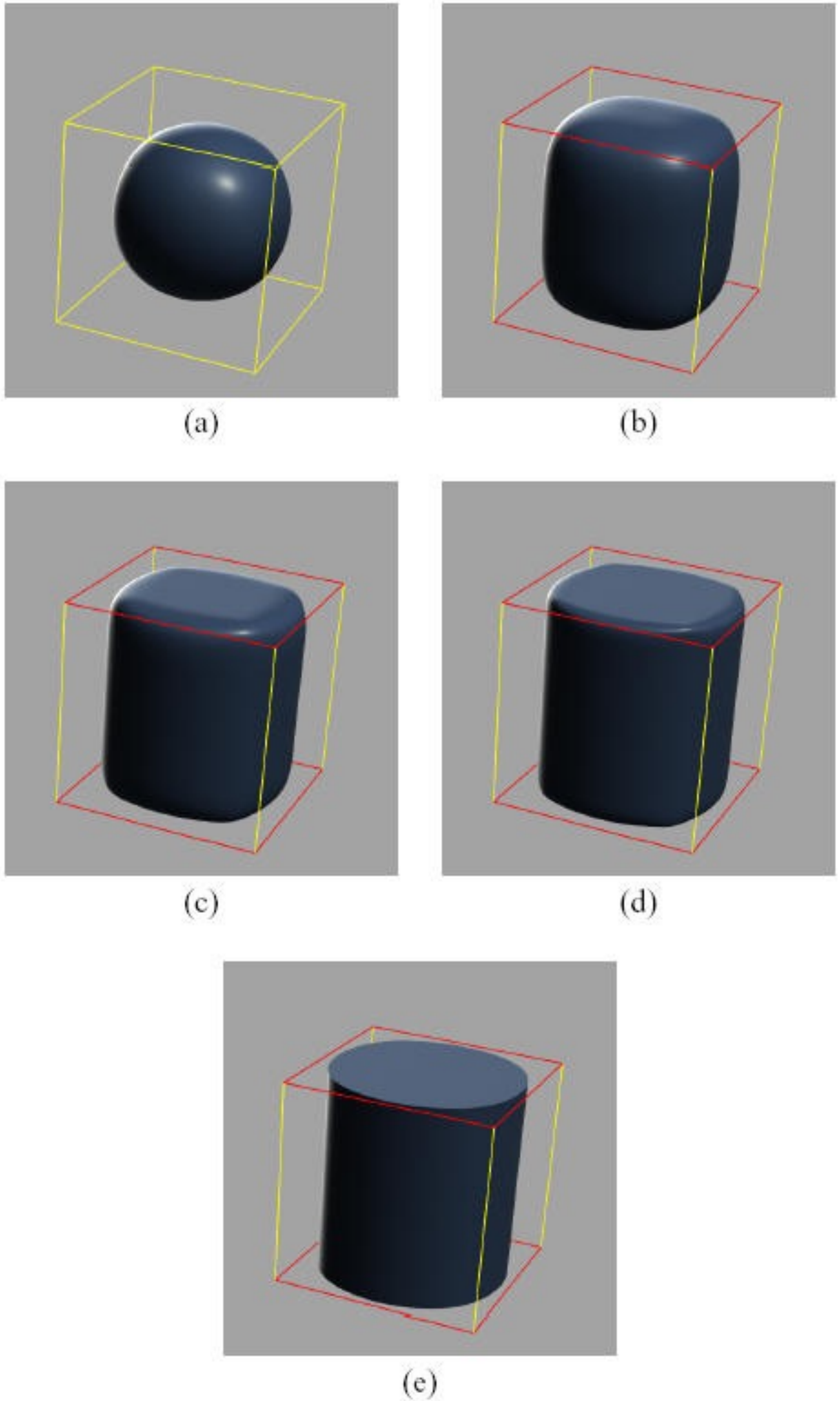
During Subdivision,

- if an edge is "sharp", use sharp subdivision rules. Newly created edges, are assigned a sharpness of n-1.
- If an edge is "not-sharp", use normal smooth subdivision rules.

IDEA: Edges with a sharpness of "n" do not get subdivided smoothly for "n" iterations of the algorithm.

•In the picture on the right, the control mesh is a unit cube

•Different sharpness applied



(a) (b) (c) (d) (e)

Sharp Rules

- FACE (unchanged)

$$f = \frac{1}{n}\sum_{1}^{n} v_i$$

- EDGE

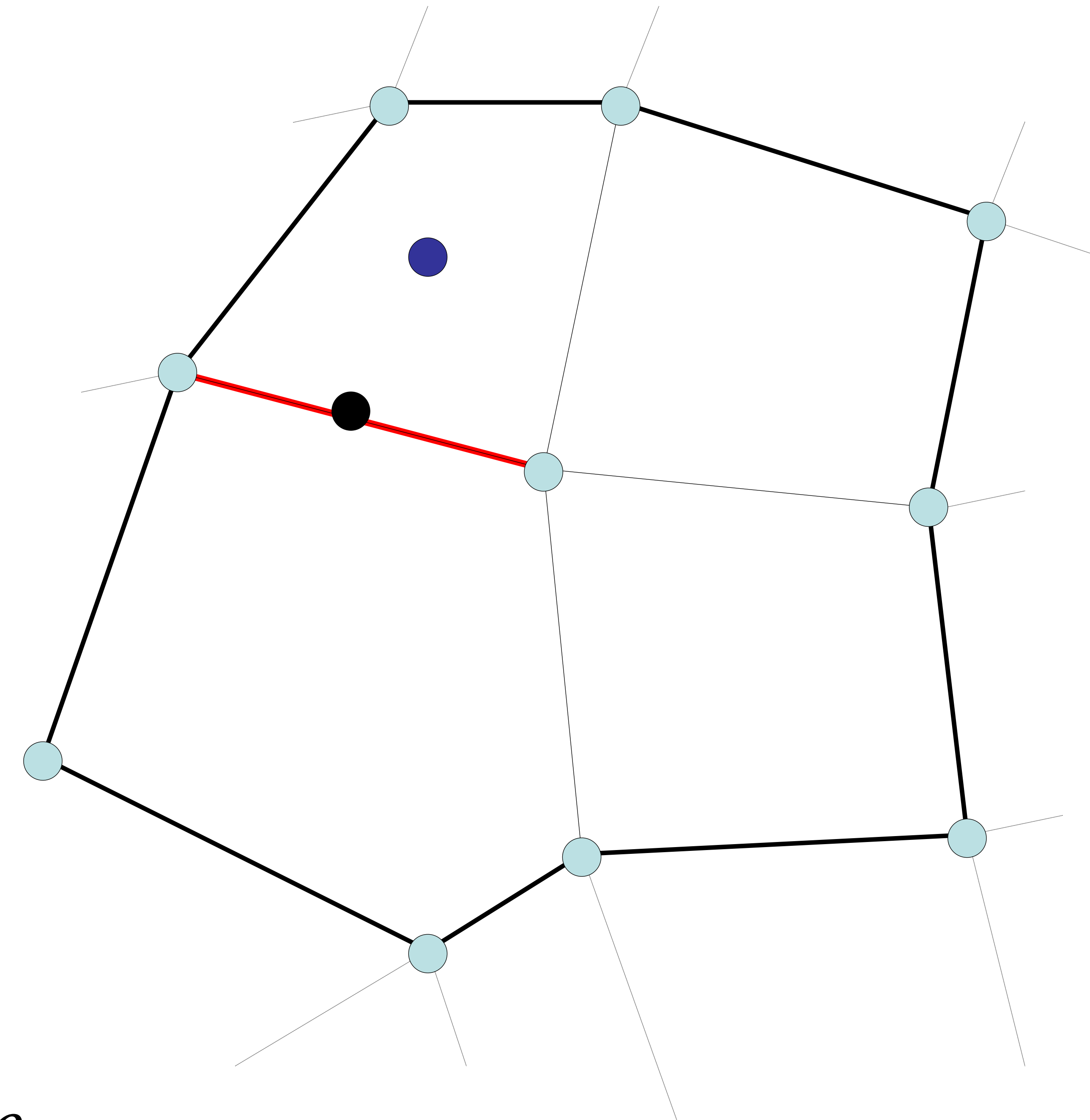
$$e = \frac{v_1 + v_2}{2}$$

VERTEX

	# adj. Sharp edges
corner	>2
crease	2

$$v_{i+1} = v_i$$

$$v_{i+1} = \frac{e_1 + 6v_i + e_2}{8}$$



Another example of creases

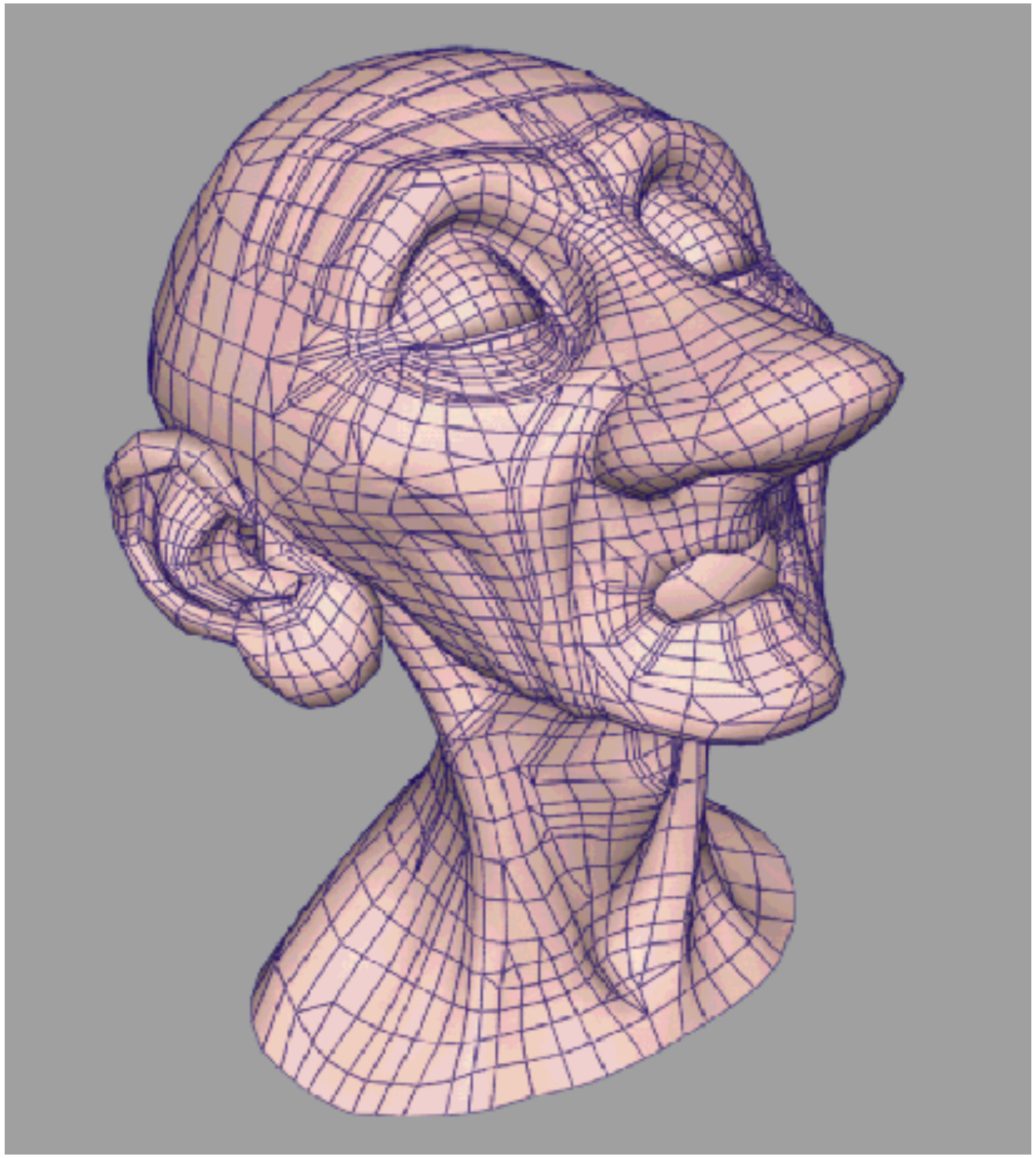
Non-Integer Sharpness

- Density of newly generated mesh increases rapidly.
- In practice, 2 or 3 iterations of subdivision is sufficient.
- Need better "control".

IDEA: Interpolate between smooth and sharp rules for non-integer sharpness values of n.

Subdivision Surfaces in character animation [DeRose98]

- Used for first time in Geri's game to overcome topological restriction of NURBS
- Modelled Geri's head, hands, jacket, trousers, shirt, tie, and shoes
- Developed cloth simulation methods



Demo movie [Geri's Game]

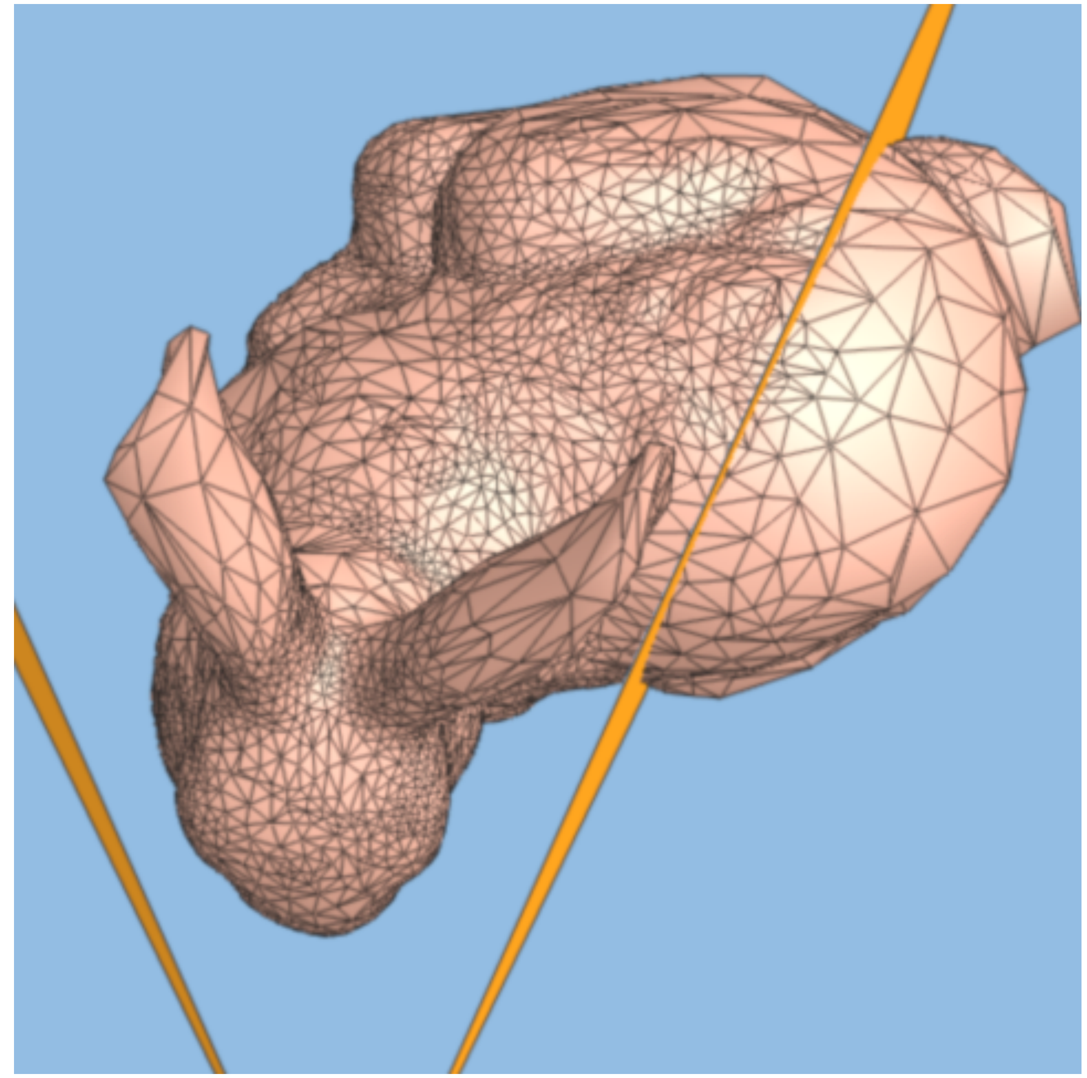
- Academy Award winning movie by Pixar

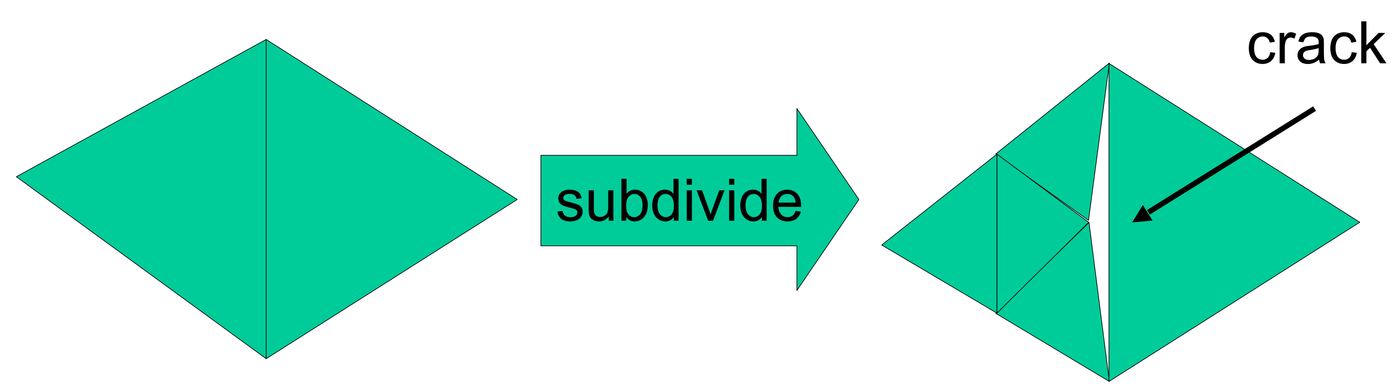
Demo of Catmull-Clark subdivision surface

- http://www.youtube.com/watch?v=lU8f0hnorU8&feature=related

Adaptive Subdivision

- Not all regions of a model need to be subdivided.
- Idea: Use some criteria and adaptively subdivide mesh where needed.
  - Curvature
  - Screen size (make triangles $<$ size of pixel)
  - View dependence
    - Distance from viewer
    - Silhouettes
    - In view frustum
  - Careful! Must ensure that “cracks” aren’t made





View-dependent refinement of progressive meshes
Hugues Hoppe.
(SIGGRAPH ’97)

Subdivision Surface Summary

- Advantages
  - Simple method for describing complex surfaces
  - Relatively easy to implement
  - Arbitrary topology
  - Local support
  - Guaranteed continuity
  - Multi-resolution
- Difficulties
  - Intuitive specification
  - Parameterization
  - Intersections

References

- A very good website for parametric curves / surfaces http://www.cs.mtu.edu/~shene/COURSES/cs3621/
- DeRose, Tony, Michael Kass, and Tien Truong. Subdivision Surfaces in Character Animation. *SIGGRAPH 98*.
- Clark, E., and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Geometric Design*, Vol. 10, No. 6, 1978.
- Doo, D. and M. Sabin. Behavior of Recursive Division Surfaces Near Extraordinary Points. Computer-Aided Design. Vol. 10, No. 6, 1978.

Links

- http://www.ibiblio.org/e-notes/Splines/basis.html
- http://i33www.ira.uka.de/applets/mocca/html/noplugin/BSplineBasis/AppBSplineBasis/index.html
- http://geom.ibds.kit.edu/applets/mocca/html/noplugin/IntBSpline/AppInsertion/index.html
- http://www.multires.caltech.edu/teaching/demos/java/chaikin.htm
- http://i33www.ira.uka.de/applets/mocca/html/noplugin/curves.html
- http://www.rose-hulman.edu/~finn/CCLI/Applets/DooSabinApplet.html