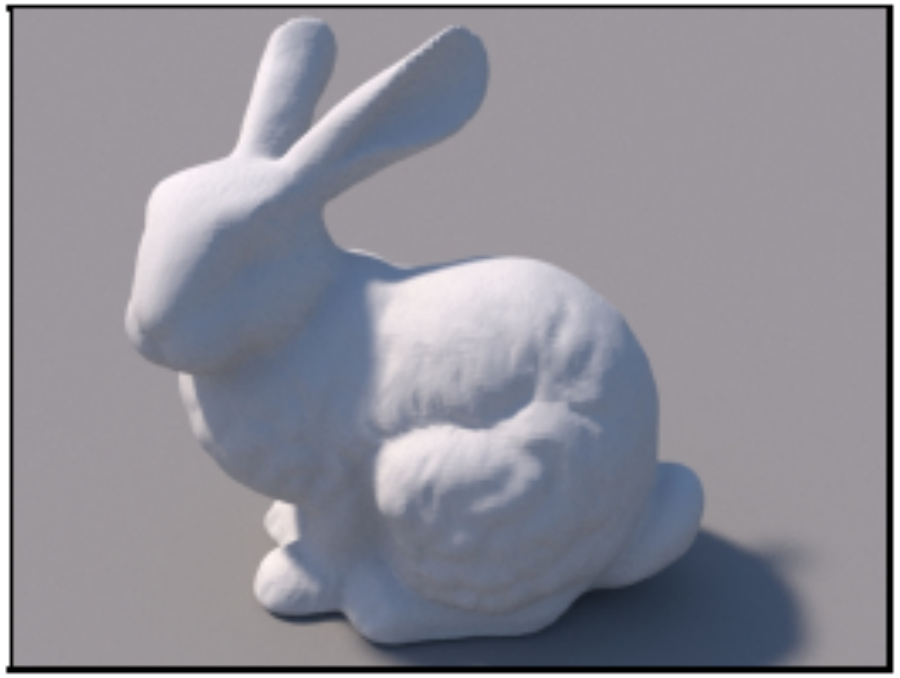
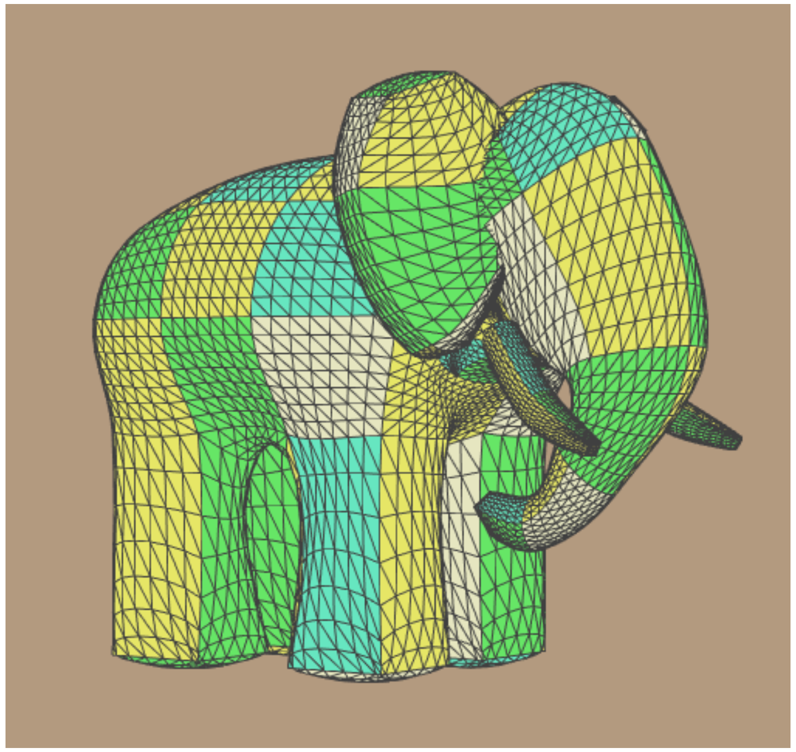
Computer Graphics 16 - Curves and Surfaces 1

Tom Thorne

Slides courtesy of Taku Komura
www.inf.ed.ac.uk/teaching/courses/cg

Characters and objects

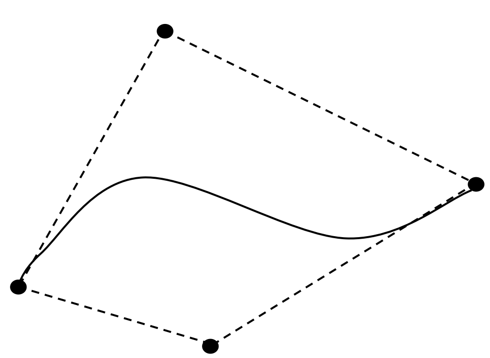
- Important for composing the scene
- Need to design and model them in the first place

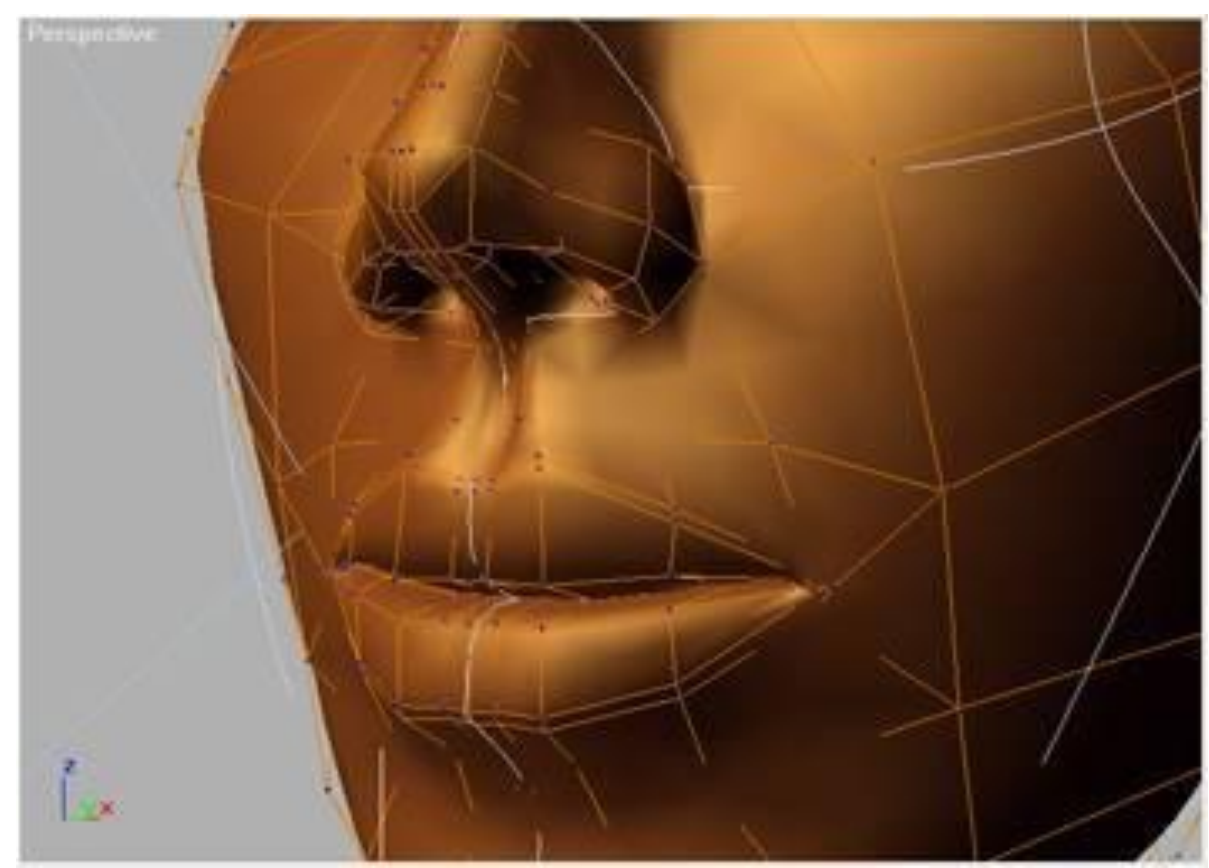




Curves and curved surfaces

Can produce smooth surfaces with less parameters

- Easier to design
- Can efficiently preserve complex structures





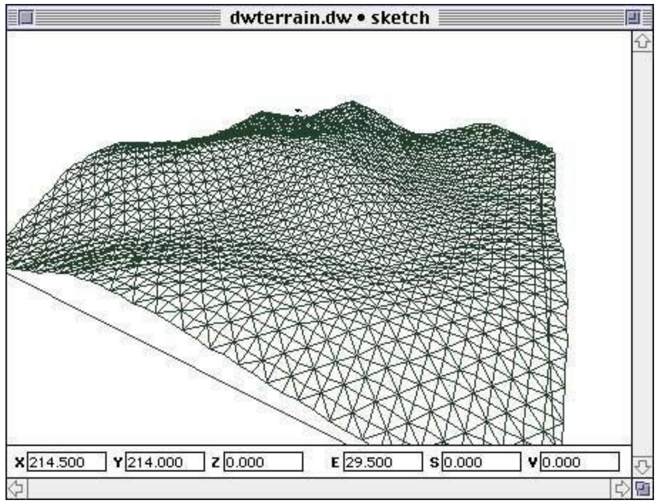
Overview

- Parametric curves
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- Bicubic patches
- Tessellation
  - Adaptive tesselation

Types of Curves and Surfaces

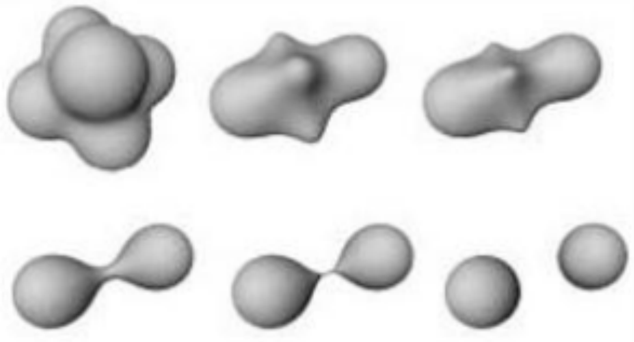
- Explicit

$$y = mx + b$$



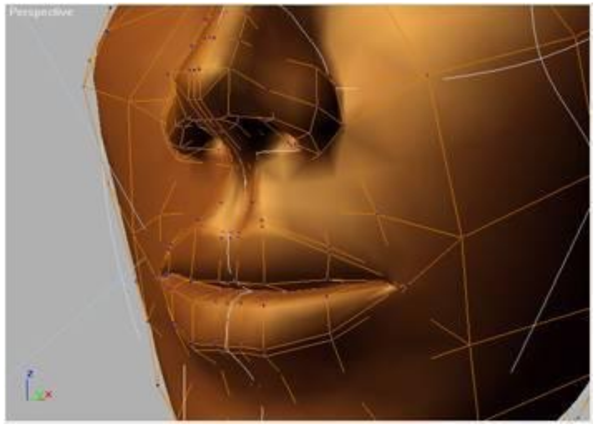
- Implicit

$$ax + by + c = 0$$



- Parametric

$$x = x_0 + (x_1 - x_0)t \qquad x = x_0 + r\cos\theta$$
$$y = y_0 + (y_t - y_0)t \qquad y = y_0 + r\sin\theta$$



Why parametric?

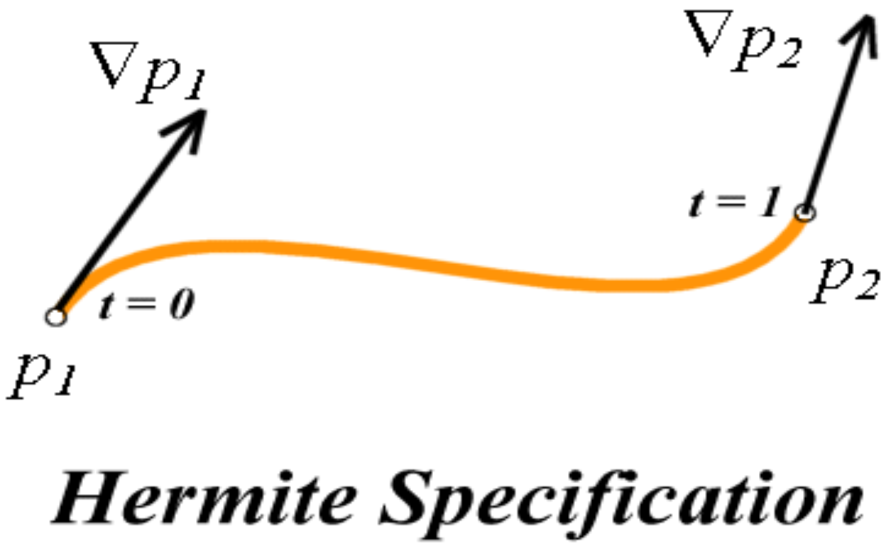
- Simple and flexible
- The function of each coordinate can be defined independently.
  - (x(t), y(t)) : 1D curve in 2D space
  - (x(t), y(t), z(t)) : 1D curve in 3D space
  - (x(s,t), y(s,t), z(s,t)) : 2D surface in 3D space
- Polynomial are suitable for creating smooth surfaces with less computation

$$x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

Overview

- **Parametric curves**
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- Bicubic patches
- Tessellation
  - Adaptive tesselation

Hermite curves

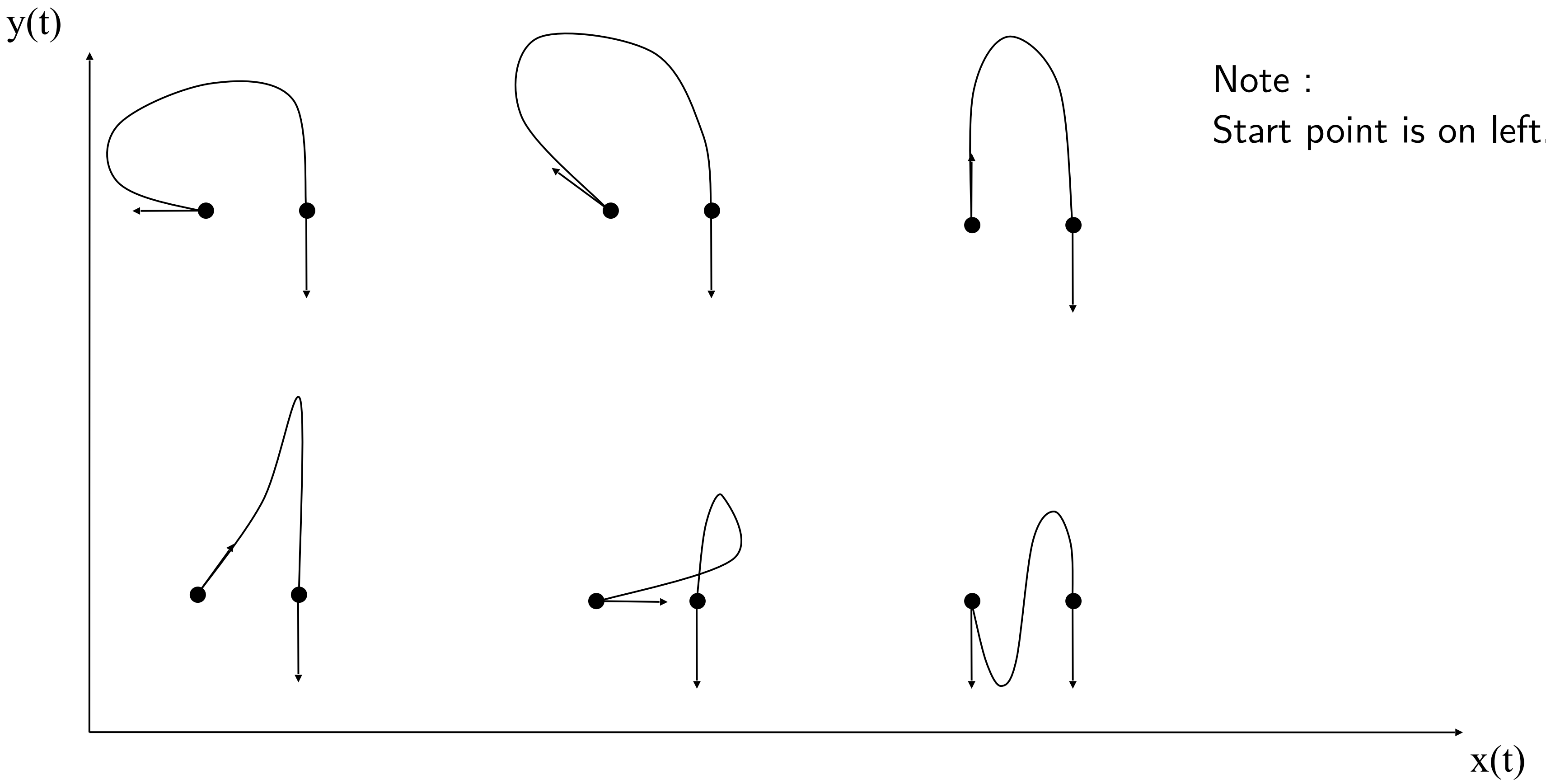


- A cubic polynomial

$$x(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

- t ranging from 0 to 1
- Polynomial can be specified by the position of, and gradient at, each endpoint of curve.

Family of Hermite curves.



Note :
Start point is on left.

Finding Hermite coefficients

Can solve them by using the boundary conditions

$$x(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 \qquad x'(t) = 3a_3 t^2 + 2a_2 t + a_1$$

- Substituting for t at each endpoint:

$$x_0 = x(0) = a_0 \qquad x'_0 = x'(0) = a_1$$

$$x_1 = x(1) = a_3 + a_2 + a_1 + a_0 \qquad x'_1 = x'(1) = 3a_3 + 2a_2 + a_1$$

- Solution is

$$a_0 = x_0 \qquad a_1 = x'_0$$

$$a_2 = -3x_0 - 2x'_0 + 3x_1 - x'_1 \qquad a_3 = 2x_0 + x'_0 - 2x_1 + x'_1$$

$$\begin{aligned} x(t) =& (2x_0 + x'_0 - 2x_1 + x'_1)t^3 + (-3x_0 - 2x'_0 + 3x_1 - x'_1)t^2 \\ &+ x'_0 t + x_0 \end{aligned}$$

The Hermite matrix

The resultant polynomial can be expressed in matrix form:

$$x(t) = t^T M_h q \qquad \text{(q is the control vector)}$$

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{bmatrix}$$

We can now define a parametric polynomial for each coordinate required independently, ie. X(t), Y(t) and Z(t)

Hermite Basis (Blending) Functions

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{bmatrix}$$

$$= \underline{\underline{(2t^3 - 3t^2 + 1)}}x_0 + \underline{\underline{(t^3 - 2t^2 + t)}}x_0' + \underline{\underline{(-2t^3 + 3t^2)}}x_1 + \underline{\underline{(t^3 - t^2)}}x_1'$$

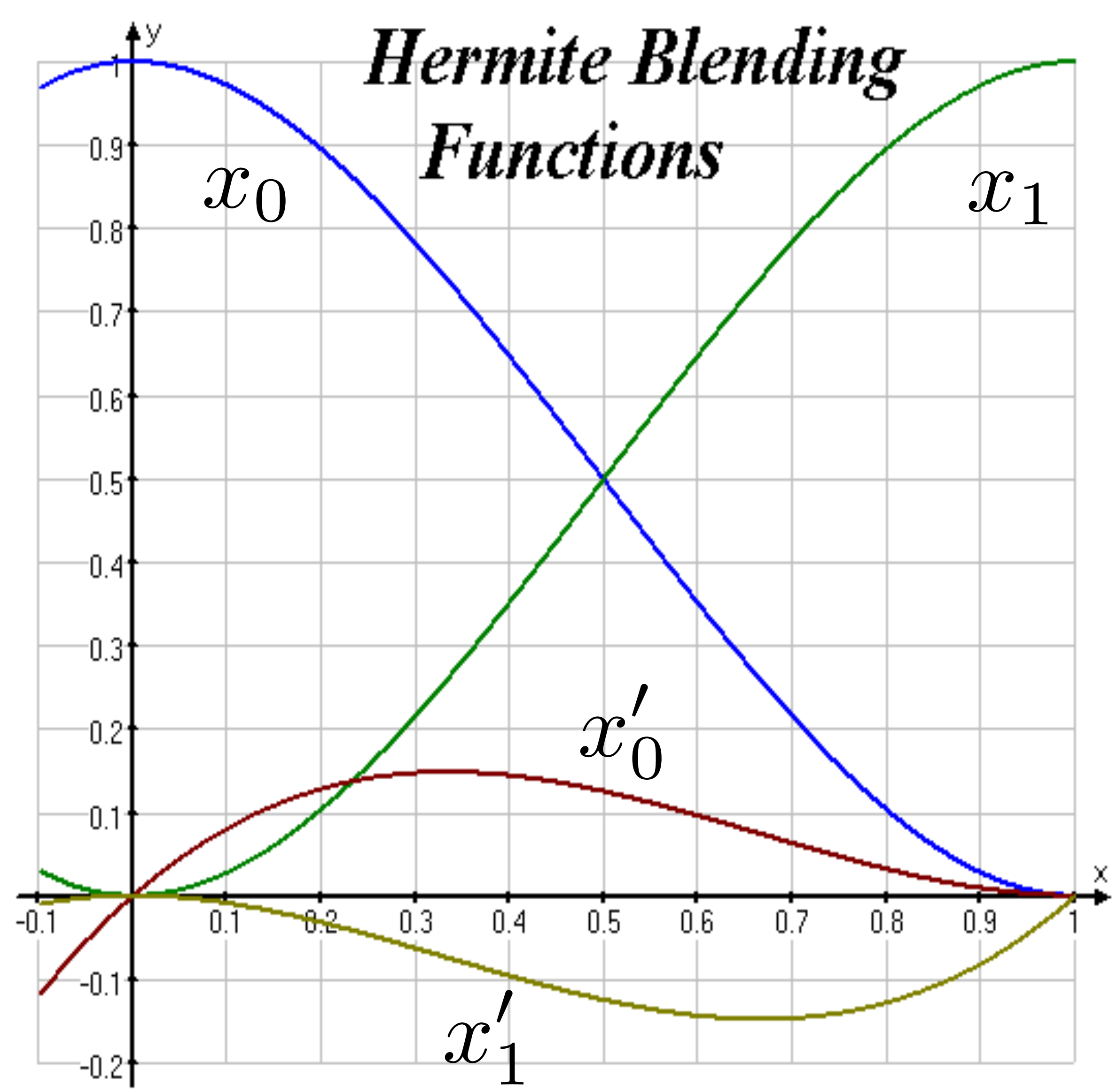
Hermite Basis (Blending) Functions

$$X(t) = \underline{\underline{(2t^3 - 3t^2 + 1)}}x_0 + \underline{\underline{(t^3 - 2t^2 + t)}}x_0' + \underline{\underline{(-2t^3 + 3t^2)}}x_1 + \underline{\underline{(t^3 - t^2)}}x_1'$$

The graph shows the shape of the four basis functions – often called blending functions.

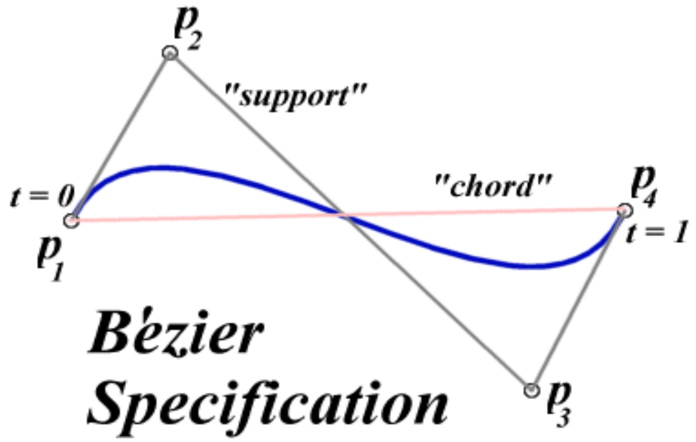
They are labelled with the elements of the control vector that they weight.

Note that at each end only position is non-zero, so the curve must touch the endpoints



Bézier Curves

- Hermite cubic curves are difficult to model – need to specify point and gradient.
- Paul de Casteljau who was working for Citroën, invented another way to compute the curves
- Publicised by Pierre Bézier from Renault
- By only giving points instead of the derivatives

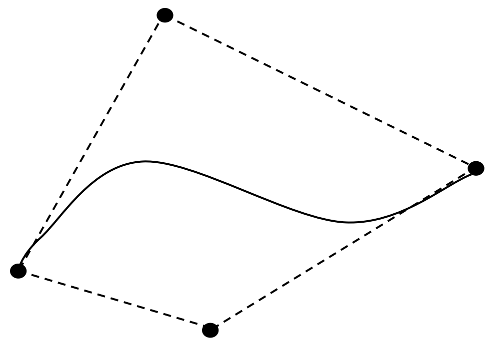


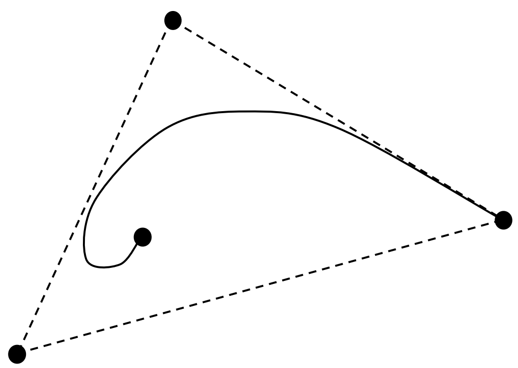
Bézier Curves

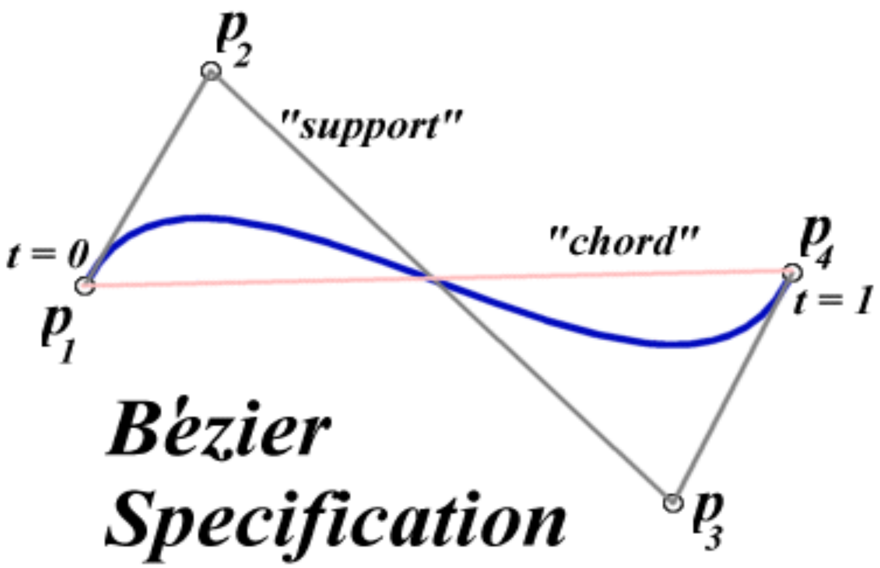
Can define a curve by specifying 2 endpoints and 2 additional control points

The two middle points are used to specify the gradient at the endpoints

Fit within the convex hull by the control points







Bézier Matrix

- The cubic form is the most popular (M is the Bézier matrix)

$$x(t) = t^T M_b q$$

- With n=4 and r=0,1,2,3 we get:

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$X(t) = (-t^3 + 3t^2 - 3t + 1)q_0 + (3t^3 - 6t^2 + 3t)q_1 + (-3t^3 + 3t^2)q_2 + (t^3)q_3$$

- Similarly for y(t) and z(t)

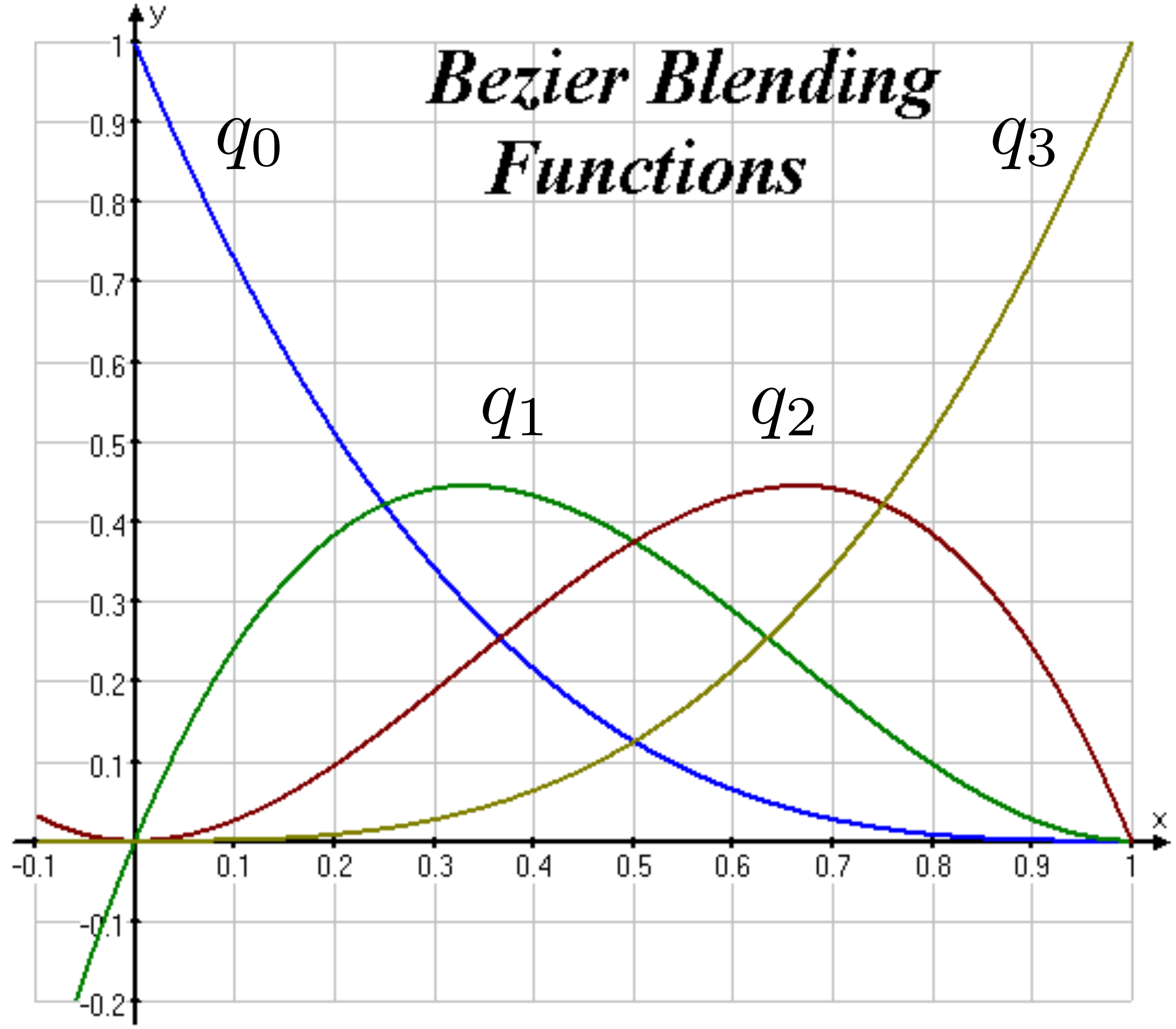
Bézier blending functions

This is how the polynomials for each coefficient look

The functions sum to 1 at any point along the curve.

Endpoints have full weight

The weights of each function is clear and the labels show the control points being weighted.

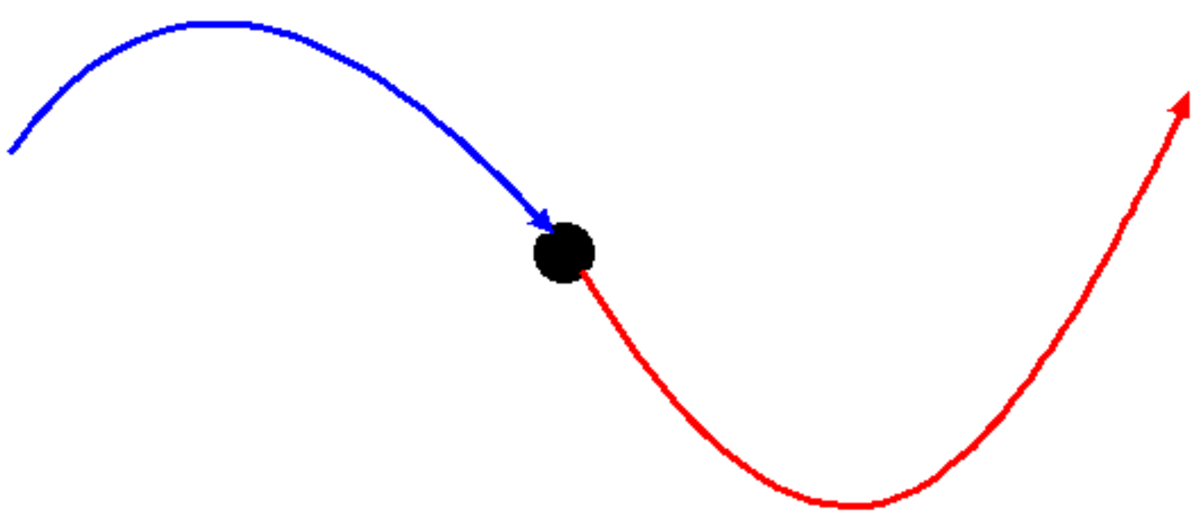


How to produce complex, long curves?

- We could only use 4 control points to design curves.
- What if we want to produce long curves with complex shapes.
- How can we do that?

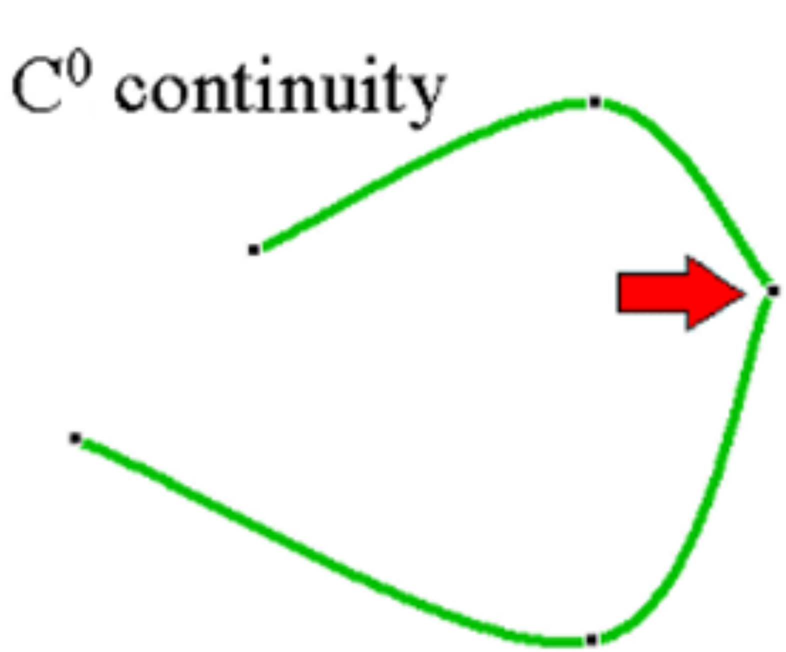
Drawing Complex Long Curves

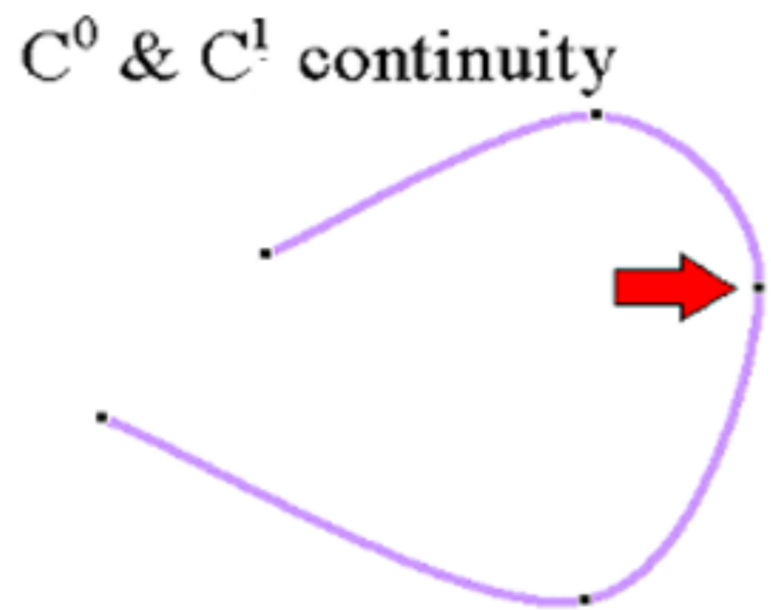
- Using higher order curves
  - costly
  - Need many multiplications
- Piece together low order curves
  - Need to make sure the connection points are smooth



Continuity between curve segments

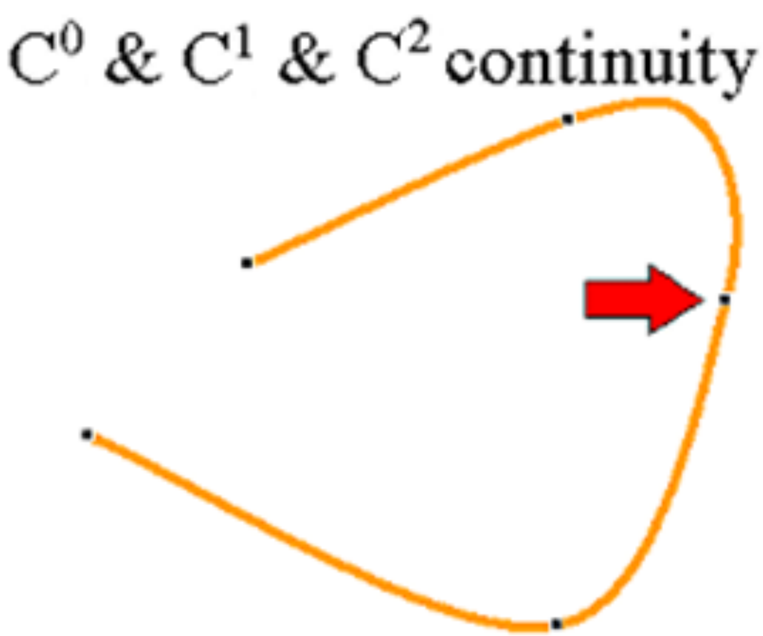
- If the direction and magnitude of $\frac{d^n X(t)}{dt^n}$ are equal at the join point, the curve is called C^ncontinuous
- i.e. if two curve segments are simply connected, the curve is C^0 continuous
- If the tangent vectors of two cubic curve segments are equal at the join point, the curve is C^1 continuous

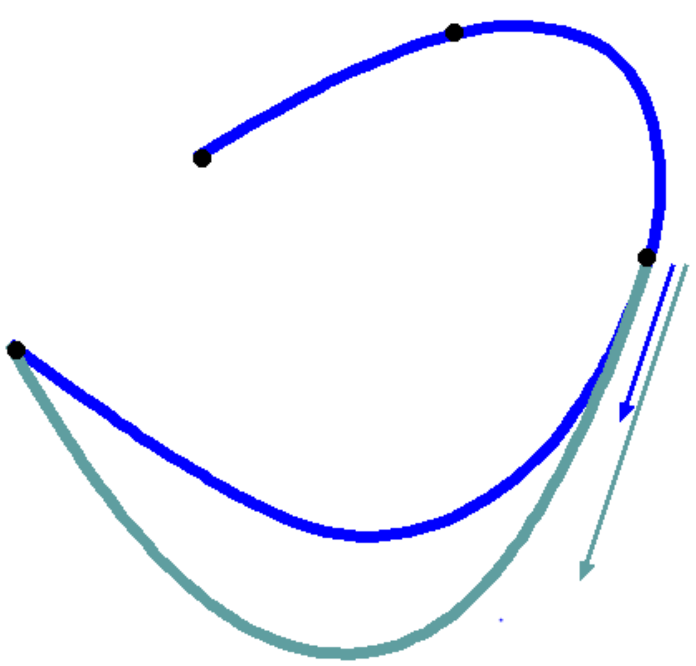




Continuity between curve segments

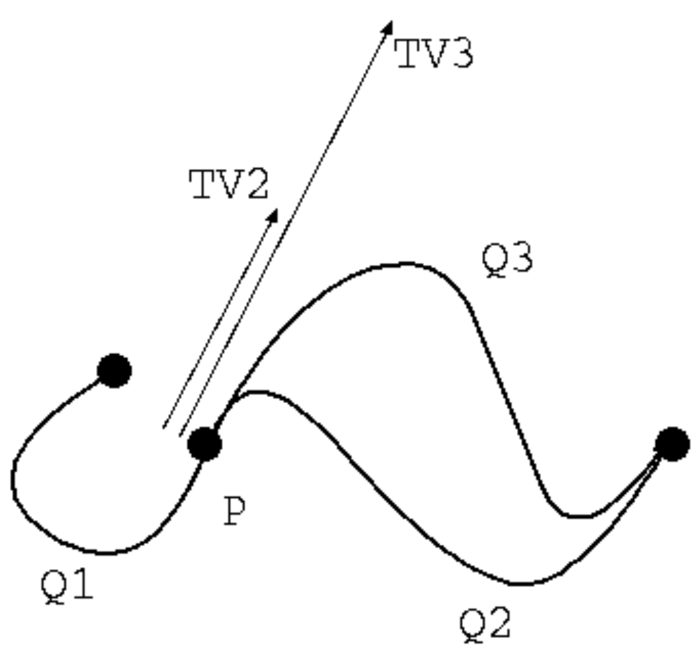
- If the directions (but not necessarily the magnitudes) of two segments' tangent vectors are equal at the join point, the curve has G1 continuity

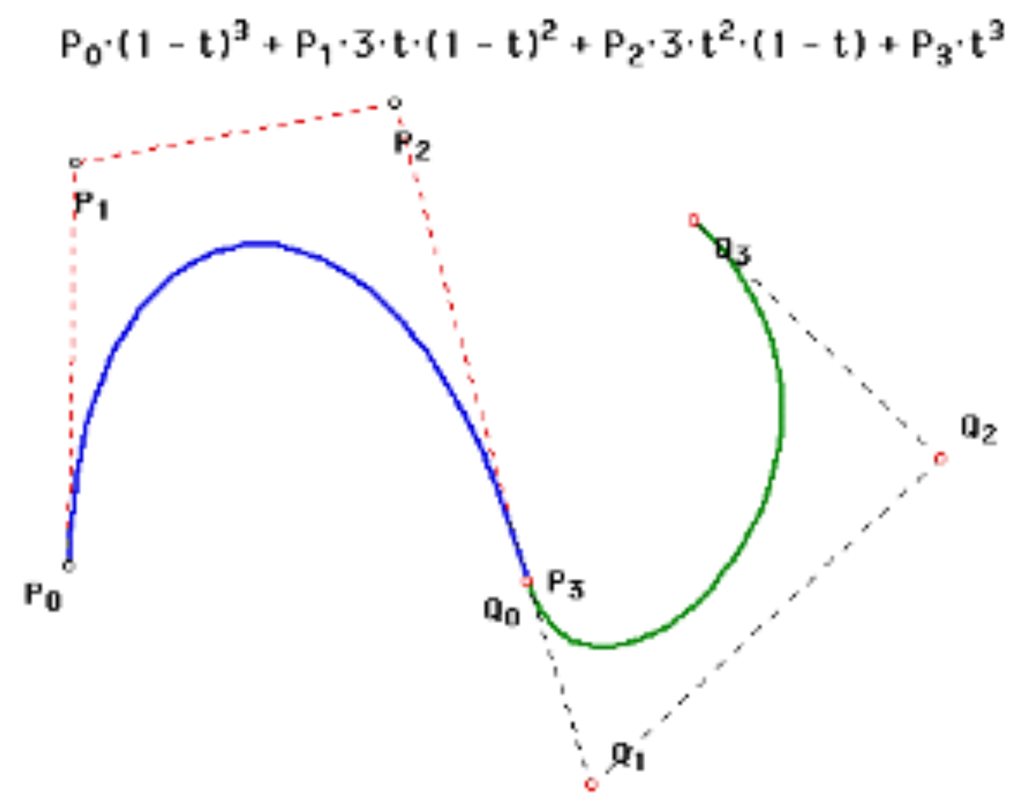




Continuity with Hermite and Bézier Curves

- How to achieve C0,C1,G1 continuity?



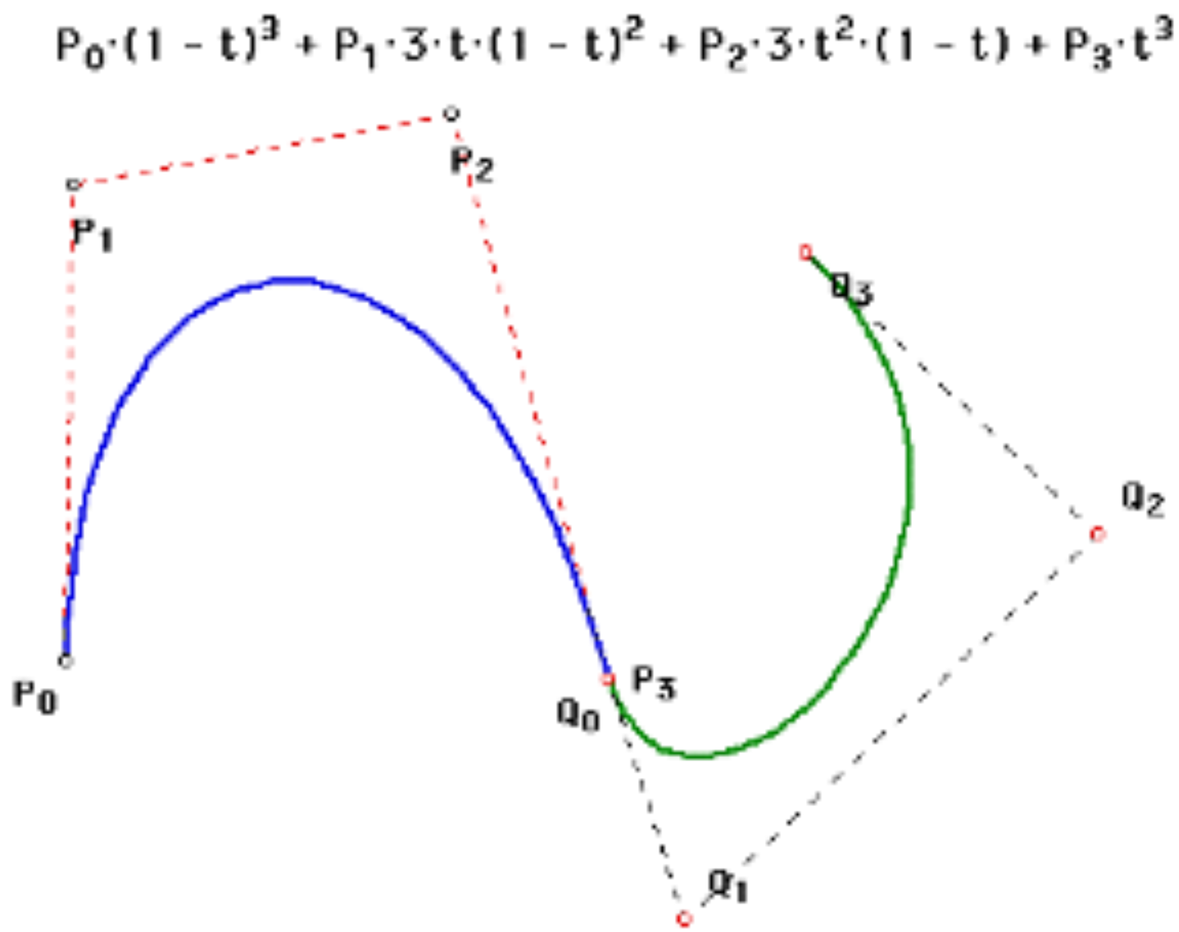


Joining Bézier Curves

- G1 continuity is provided at the endpoint when

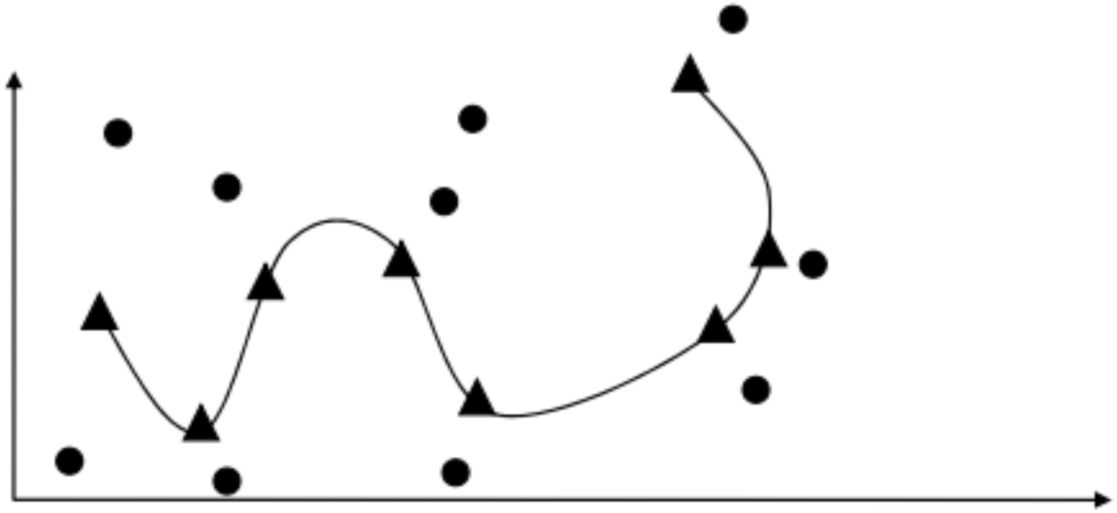
$$p_3 - p_2 = k(q_1 - q_0)$$

- if k=1, C1 continuity is obtained



Uniform cubic B-splines

- Another popular form of curve
- The curve does not necessarily pass through the control points
- Can produce a longer continuous curve without worrying about the boundaries
- Has C2 continuity at the boundaries



Uniform cubic B-splines

- The matrix form and the basis functions
- The knots specify the range of the curve

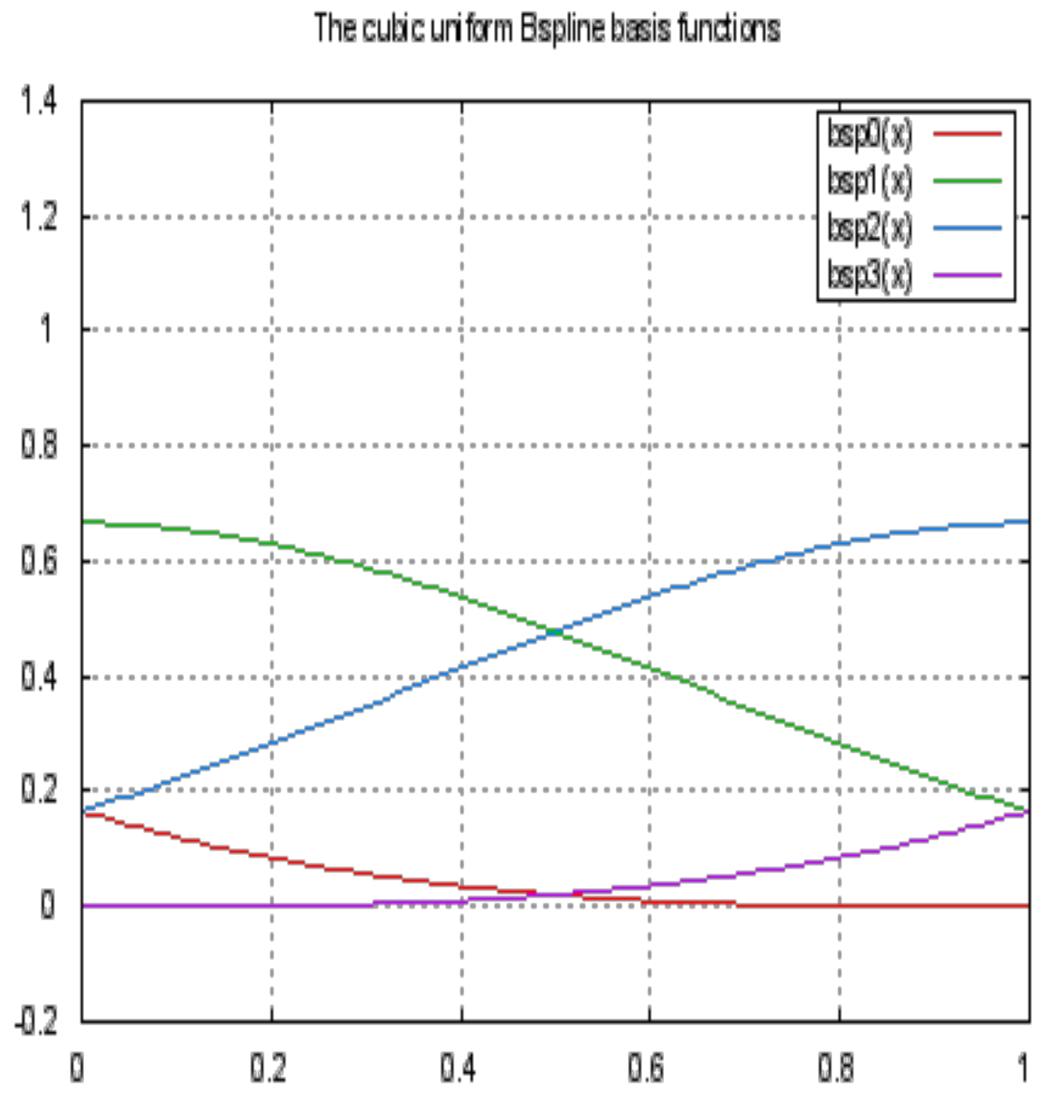
$$X(t) = \mathbf{t}^{\mathrm{T}}\mathbf{M}\mathbf{Q}^{(\mathrm{i})} \qquad for \quad t_i \le t \le t_{i+1}$$

$$where \qquad \mathbf{Q}^{(\mathrm{i})} = (x_{i-3}, \ldots, x_i)$$

$$\mathbf{M} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix},$$

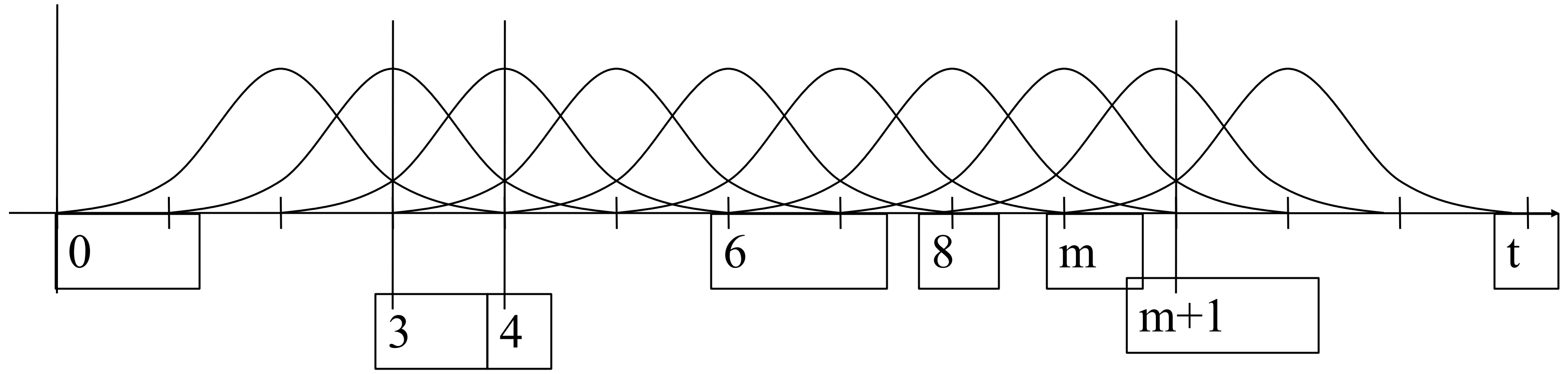
$$\mathbf{t}^{\mathrm{T}} = \left((t-t_i)^3, (t-t_i)^2, t-t_i, 1\right)$$

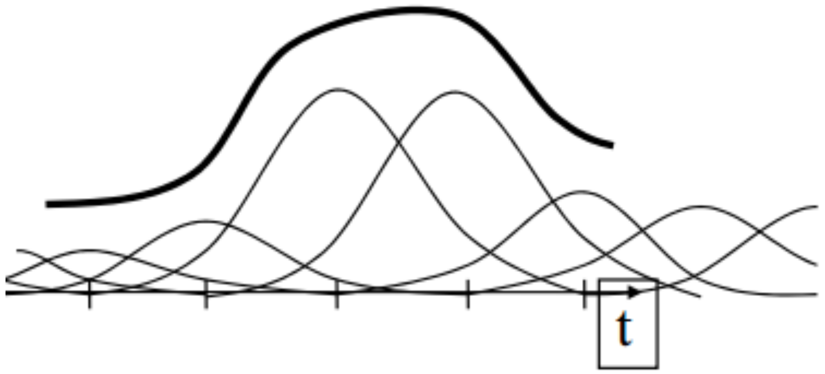
$$t_i : \text{knots}, \; 3 \le i$$



Uniform cubic B-splines

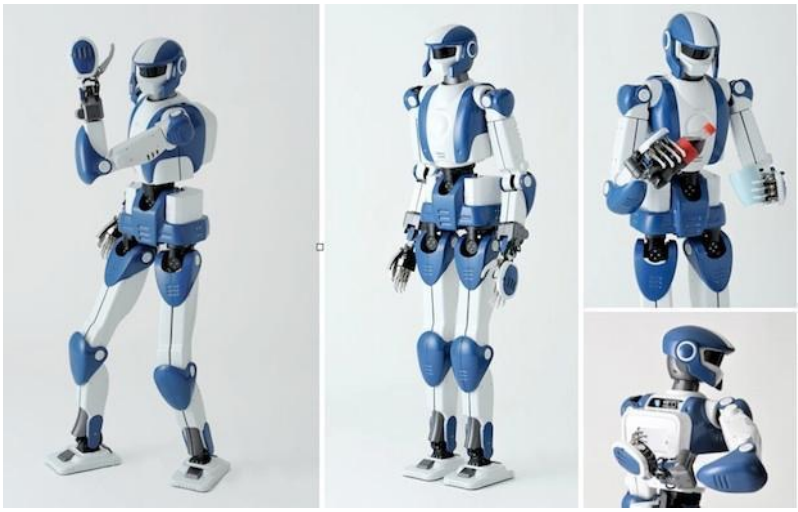
- This is how the basis splines look over the domain
- The initial part is defined after passing the fourth knot





Another usage of uniform cubic B-splines

- Representing the joint angle trajectories of characters and robots
- Need more control points to represent a longer continuous movement
- Need C2 continuity to make the acceleration smooth
- And not changing the torques suddenly



Catmull-Rom Spline

- A curve that interpolates control points
- C1 continuous
- The tangent vectors at the endpoints of a Hermite curve are set such that they are decided by the two surrounding control points

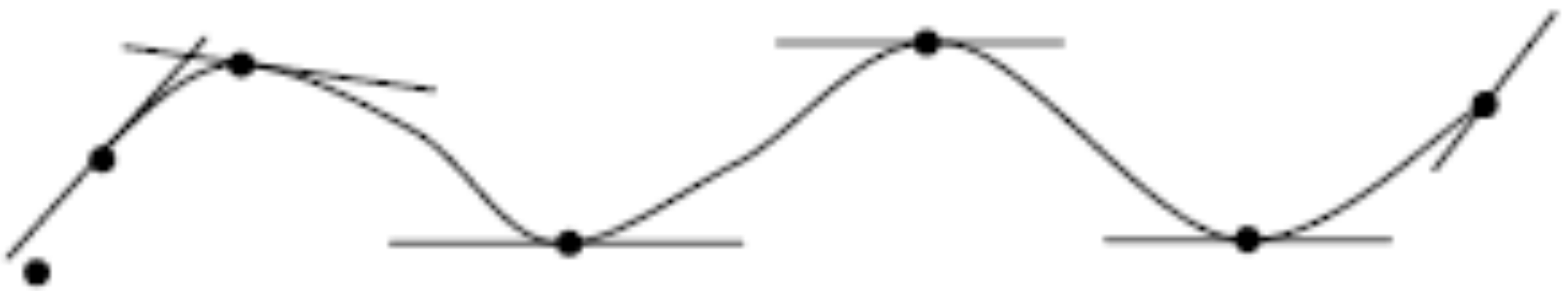
Hermite Specification

Catmull-Rom Spline

- C1 continuity

$$P^i(t) = T \cdot M_{CR} \cdot G_B$$

$$= \frac{1}{2} \cdot T \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



Overview

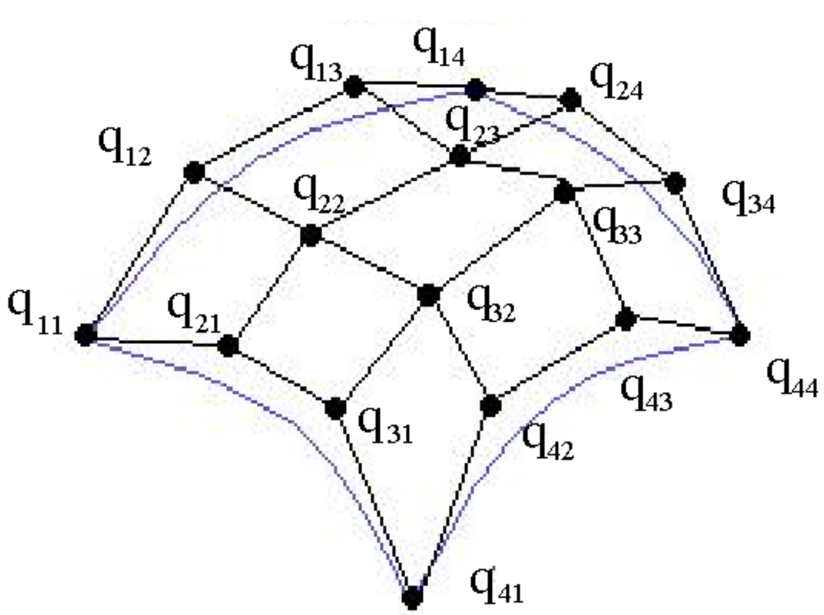
- Parametric curves
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- **Bicubic patches**
- Tessellation
  - Adaptive tesselation

Bicubic patches

- The concept of parametric curves can be extended to surfaces
- The cubic parametric curve is in the form of

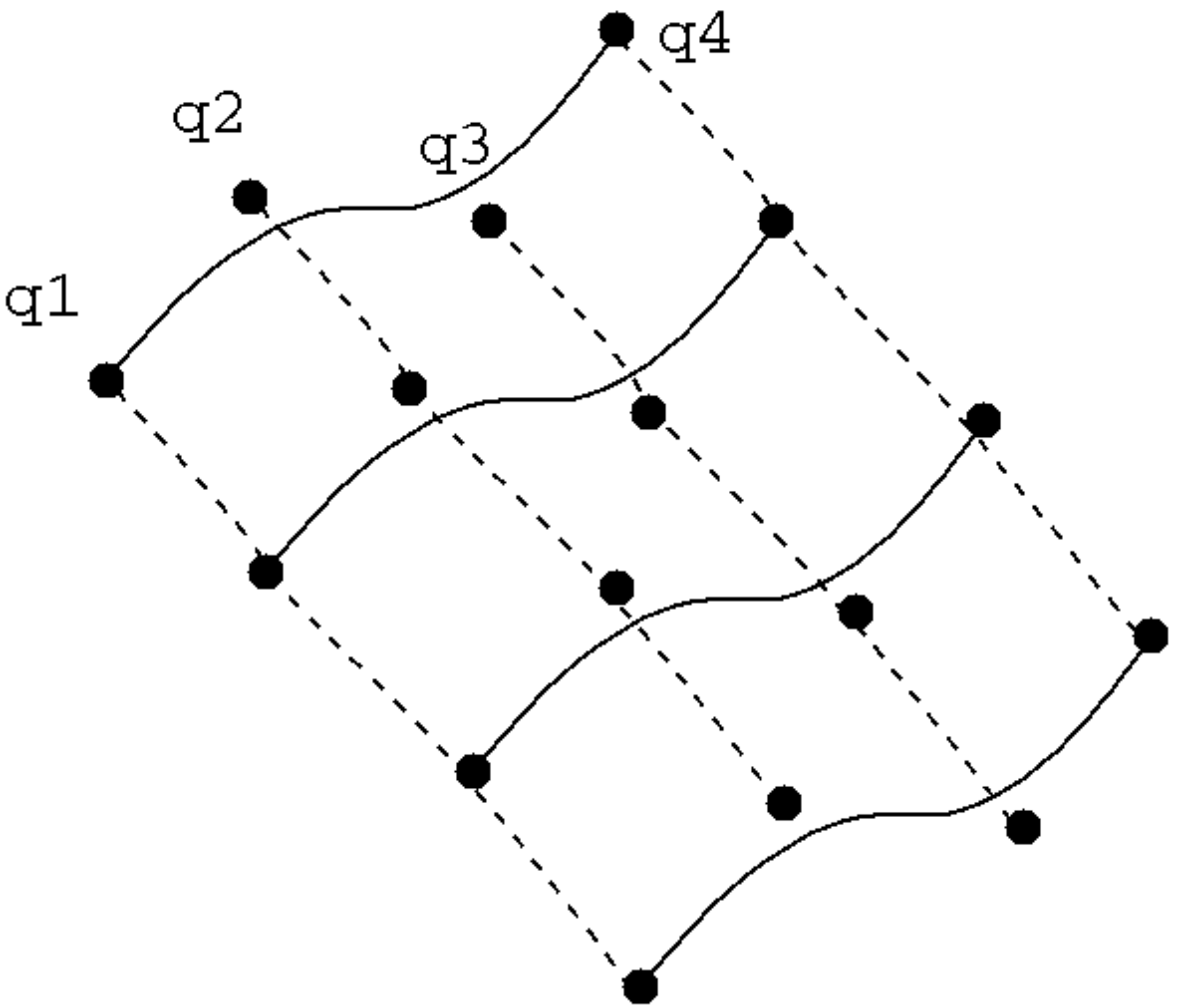
$$q(t) = t^T M q$$

- where $q = (q_1, q_2, q_3, q_4)$ control points, M is the basis matrix (Hermite or Bezier,...), $t^T = (t^3, t^2, t, 1)$



Bicubic patches

- Now we assume q_i to vary along a parameter s,
- $Q_i(s,t) = t^T M[q_1(s), q_2(s), q_3(s), q_4(s)]$
- $q_i(s)$ are themselves cubic curves
- Bicubic patch has degree 6



Bicubic patches

$$Q(s,t) = t^T.M.(s^T.M.[\mathbf{q}_{11},\mathbf{q}_{12},\mathbf{q}_{13},\mathbf{q}_{14}],\ldots,s^T.M.[\mathbf{q}_{41},\mathbf{q}_{42},\mathbf{q}_{43},\mathbf{q}_{44}])$$

$$= t^T.M.\mathbf{q}.M^T.s$$

where **q** is a 4x4 matrix $\begin{bmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ q_{12} & q_{22} & q_{32} & q_{42} \\ q_{13} & q_{23} & q_{33} & q_{43} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix}$

Each column contains the control points of

$q_1(s),\ldots,q_4(s)$

x,y,z computed by

$$x(s,t) = t^T.M.\mathbf{q}_x.M^T.s$$

$$y(s,t) = t^T.M.\mathbf{q}_y.M^T.s$$

$$z(s,t) = t^T.M.\mathbf{q}_z.M^T.s$$

Bézier example

- We compute (x,y,z) by

$$x(s,t) = t^T . M_B . q_x . M_B^T . s$$

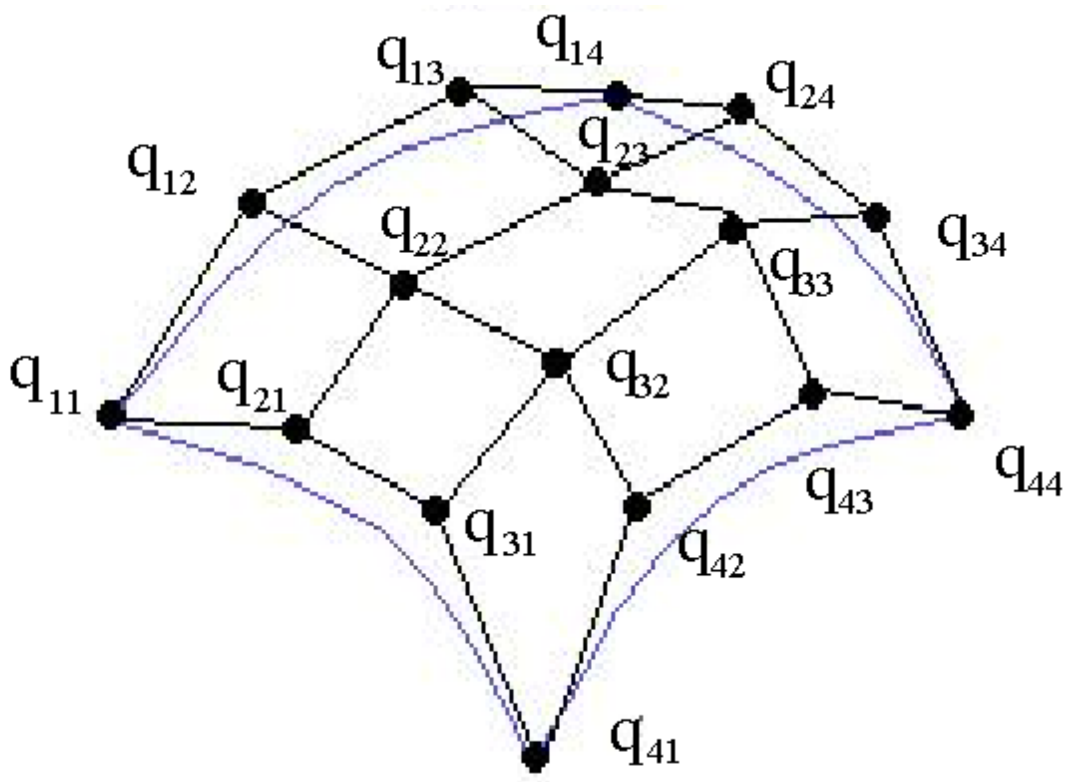
q_x *is* 4×4 *array of x coords*

$$y(s,t) = t^T . M_B . q_y . M_B^T . s$$

q_y *is* 4×4 *array of y coords*

$$z(s,t) = t^T . M_B . q_z . M_B^T . s$$

q_z *is* 4×4 *array of z coords*



Today

- Parametric curves
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- Bicubic patches
- **Tessellation**
  - Adaptive tesselation

Displaying Bicubic patches.

- Directly rasterising bicubic patches is not so easy
- Need to convert the bicubic patches into a polygon mesh
  - tessellation
- Need to compute the normals
  - vector cross product of the 2 tangent vectors.

$$\frac{\partial}{\partial s}Q(s,t) = \frac{\partial}{\partial s}(t^T.M.q.M^T.s) = t^T.M.q.M^T.\frac{\partial}{\partial s}(s)$$
$$= t^T.M.q_x.M^T.[3s^2,2s,1,0]^T$$

$$\frac{\partial}{\partial t}Q(s,t) = \frac{\partial}{\partial t}(t^T.M.q.M^T.s) = \frac{\partial}{\partial t}(t^T).M.q.M^T.s$$
$$= [3t^2,2t,1,0]^T.M.q.M^T.s$$

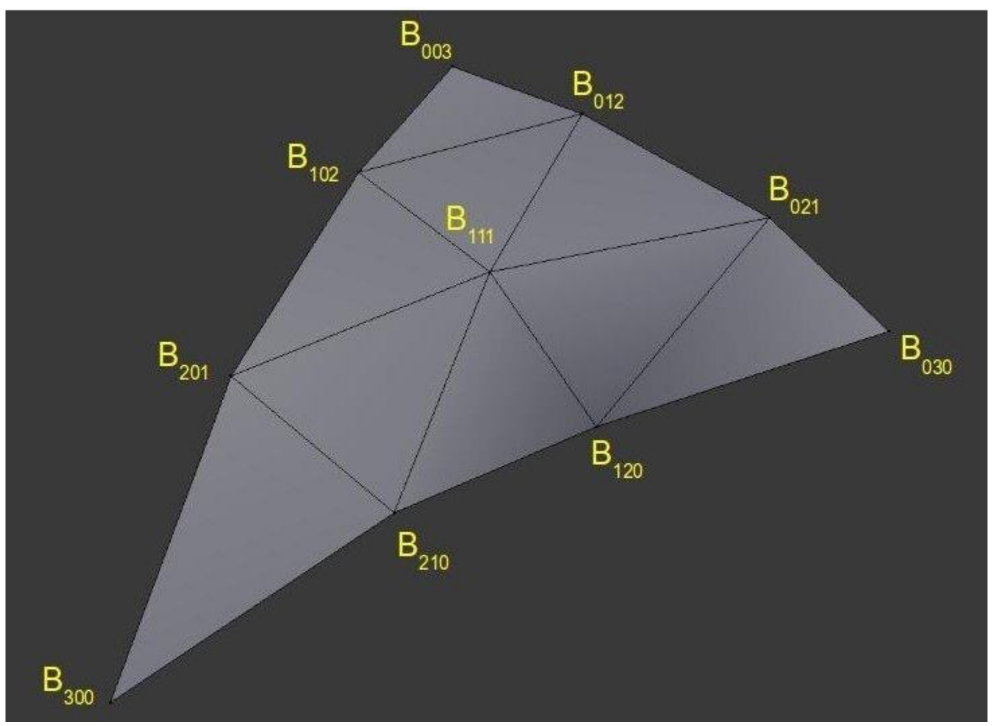
$$\frac{\partial}{\partial s}Q(s,t) \times \frac{\partial}{\partial t}Q(s,t) = (y_s z_t - y_t z_s, z_s x_t - z_t x_s, x_s y_t - x_t y_s)$$

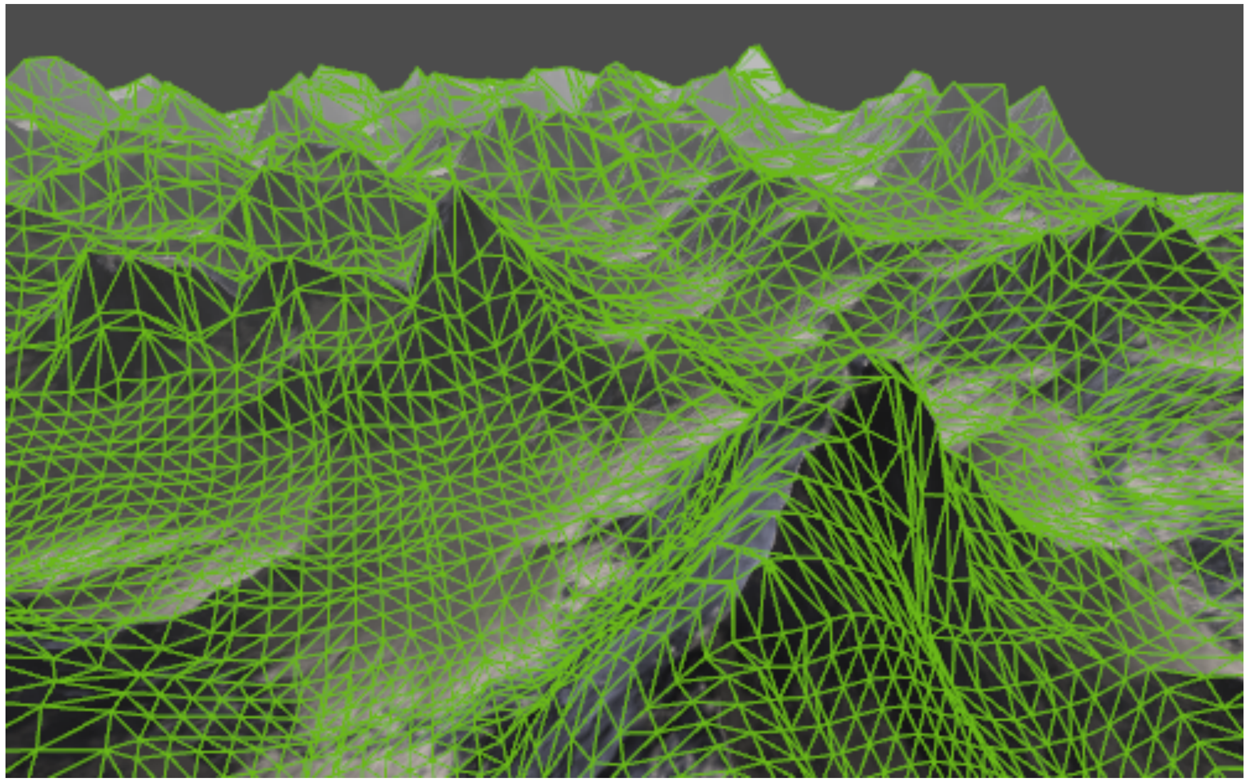
Tangent vectors can be computed by computing the partial derivatives

Then computing the cross product of the two partial derivative vectors

Tessellation

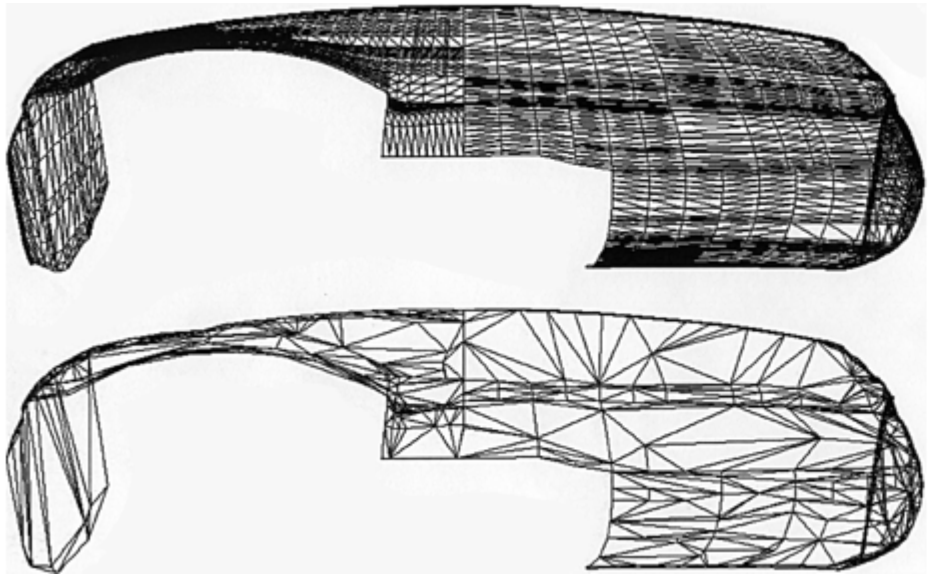
- As computers are optimised for rendering triangles, the easiest way to display parametric surfaces is to convert them into triangle meshes
- The simplest way is to do uniform tessellation, which samples points uniformly in the parameter space





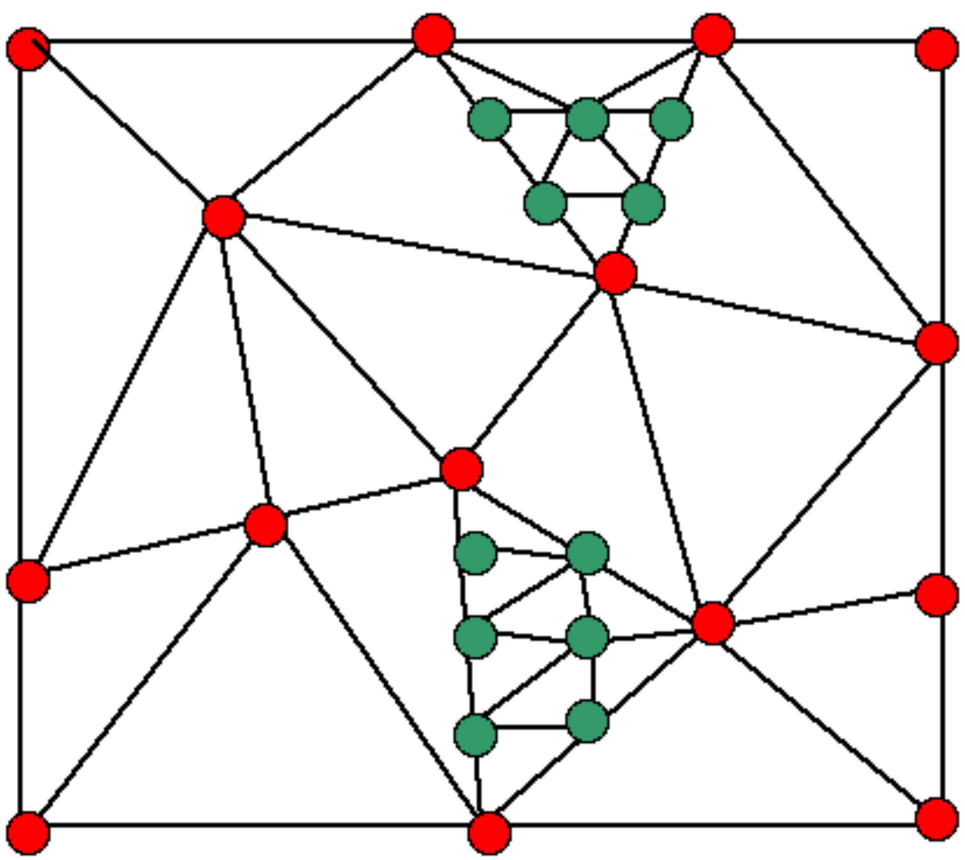
Uniform Tessellation

- Sampling points uniformly with the parameters
- What are the problems with uniform tessellation?
- Which area needs more tessellation?
- Which area does not need much tessellation?



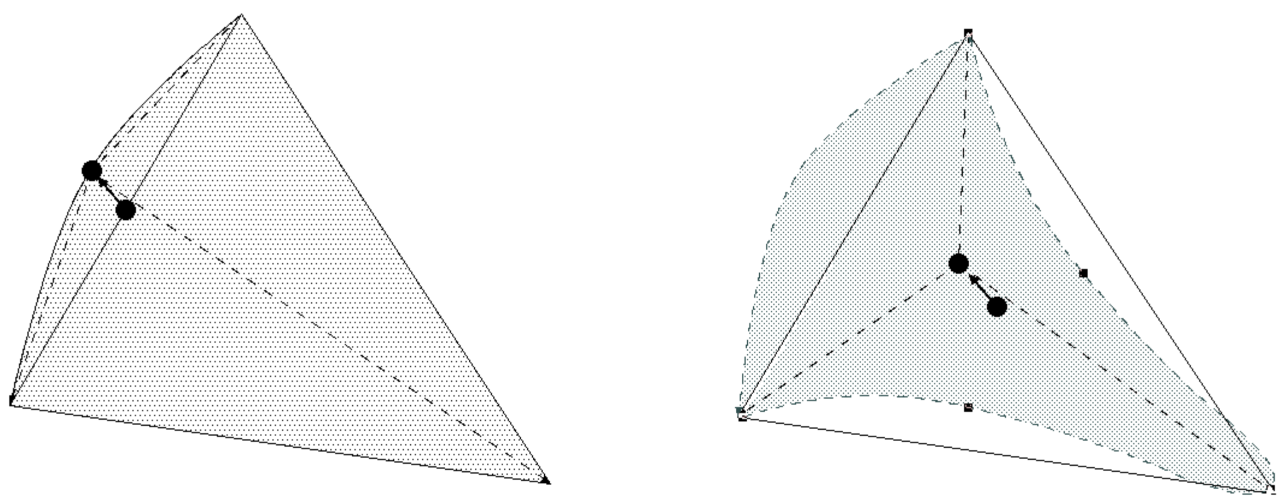
Adaptive Tessellation

- Adaptive tessellation – adapt the size of triangles to the shape of the surface



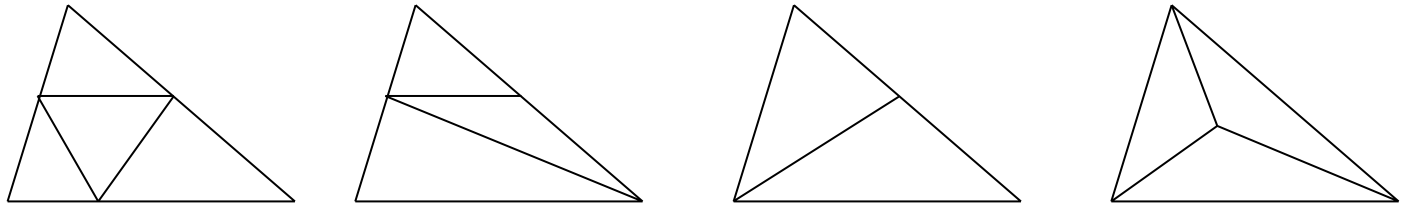
Adaptive Tessellation

- For every triangle edges, check if each edge is tessellated enough (curveTessEnough())
- If all edges are tessellated enough, check if the whole triangle is tessellated enough as a whole (triTessEnough())
- If one or more of the edges or the triangle's interior is not tessellated enough, then further tessellation is needed



Adaptive Tessellation

- When an edge is not tessellated enough, a point is created halfway between the edge points' uv-values
- New triangles are created and the tessellator is once again called with the new triangles as input



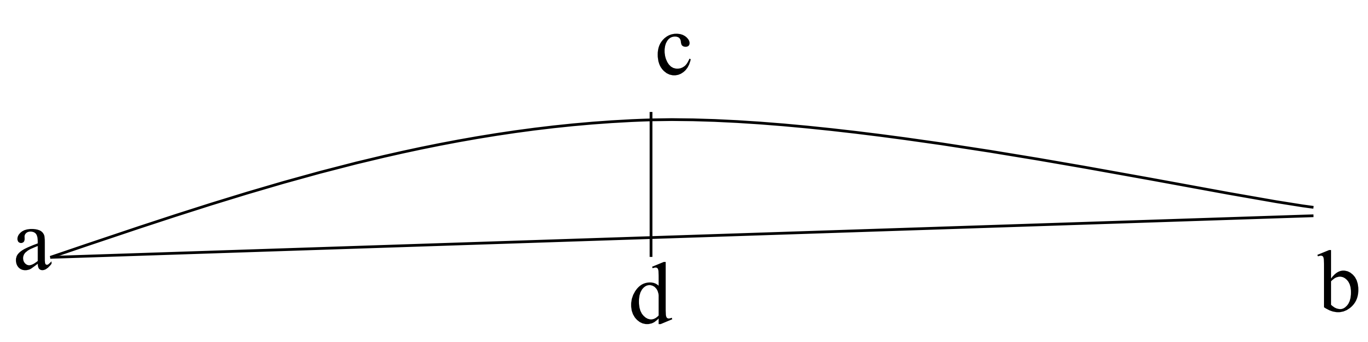
Four cases of further tessellation

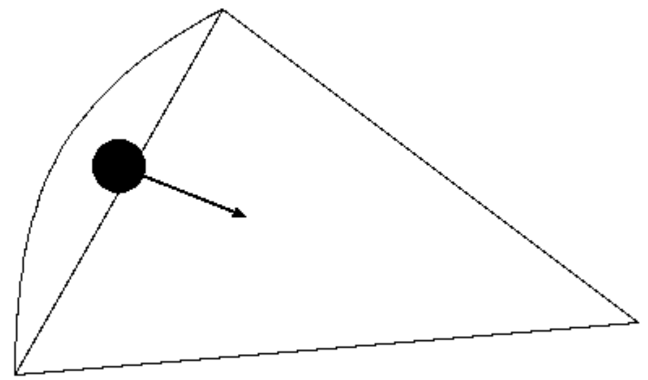
Adaptive Tessellation

```
AdaptiveTesselate(p,q,r)

•  tessPQ=not curveTessEnough(p,q)
•  tessQR=not curveTessEnough(q,r)
•  tessRP=not curveTessEnough(r,p)

•  If (tessPQ and tessQR and tessRP)
   - AdaptiveTessellate(p,(p+q)/2,(p+r)/2);
   - AdaptiveTessellate(q,(q+r)/2,(q+p)/2);
   - AdaptiveTessellate(r,(r+p)/2,(r+q)/2);
   - AdaptiveTessellate((p+q)/2,(q+r)/2,(r+p)/2);

•  else if (tessPQ and tessQR)
   - AdaptiveTessellate(p,(p+q)/2,r);
   - AdaptiveTessellate((p+q)/2,(q+r)/2,r);
   - AdaptiveTessellate((p+q)/2,q,(q+r)/2);

•  else if (tessPQ)
   - AdaptiveTessellate(p,(p+q)/2,r);
   - AdaptiveTessellate(q,r,(p+q)/2);

•  else if (not triTessEnough(p,q,r))
   - AdaptiveTessellate((p+q+r)/3,p,q);
   - AdaptiveTessellate((p+q+r)/3,q,r);
   - AdaptiveTessellate((p+q+r)/3,r,p);
end;
```

curveTessEnough

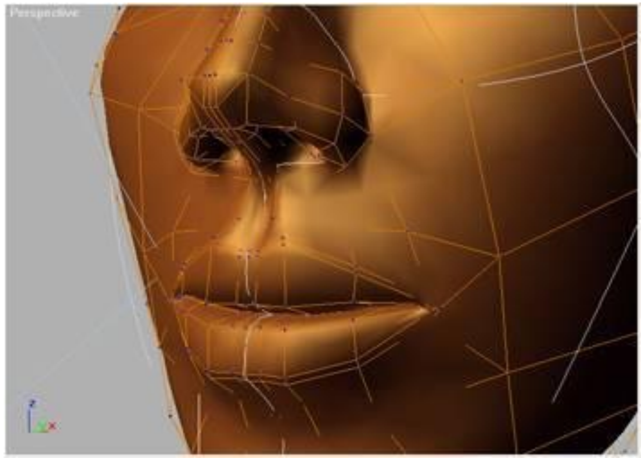
- Say you are to judge whether ab needs tessellation
- You can compute the midpoint c, and compute the curve's distance l from d, the midpoint of ab
- Check if l/||a-b|| is under a threshold
- Can do something similar for triTessEnough
  - Sample at the mass center and calculate its distance from the triangle





On-the-fly tessellation

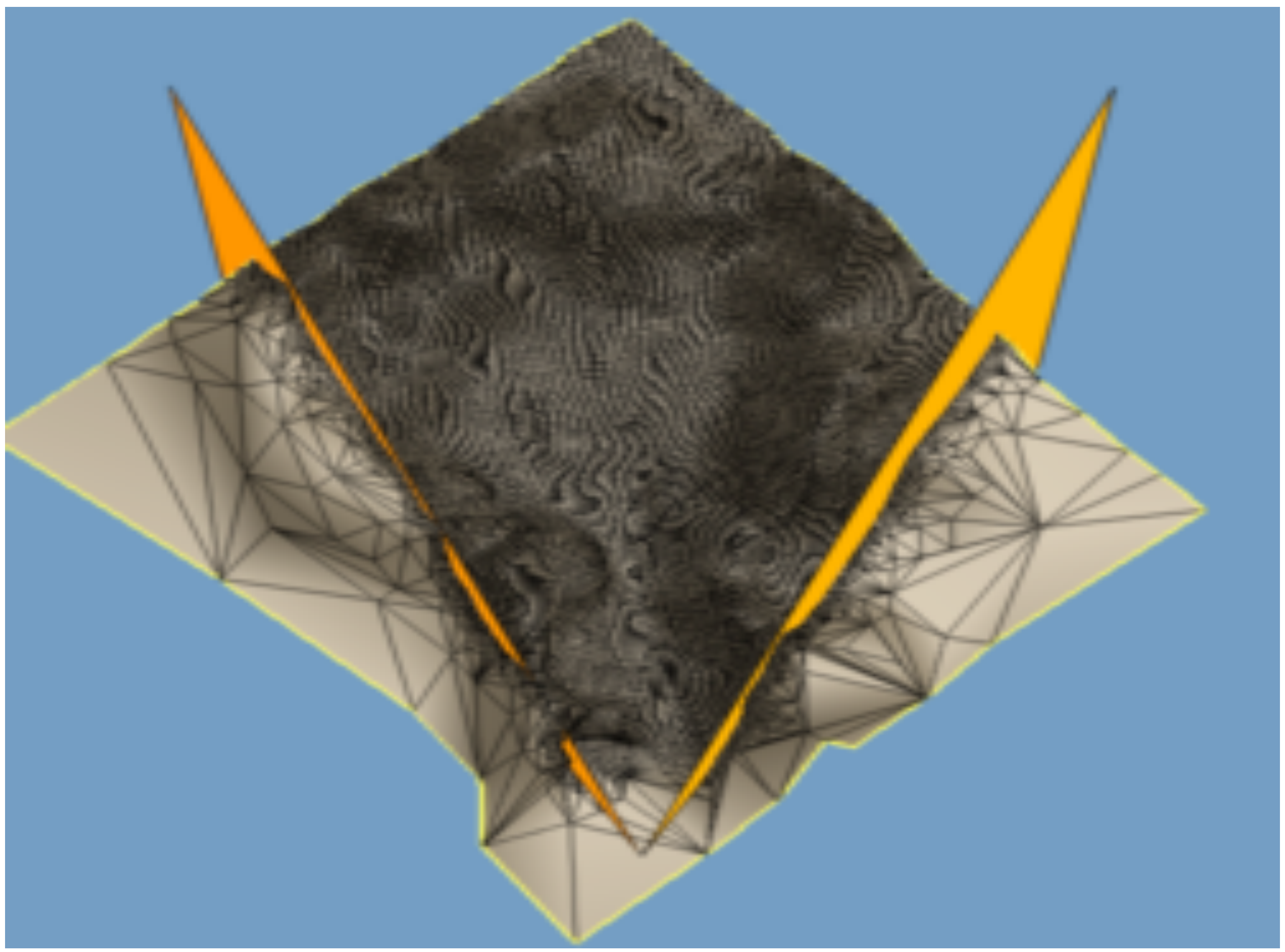
- In many cases, it is preferred to tessellate on-the-fly
- The size of the data can be kept small
- Tessellation is a highly parallel process
  - Can make use of the GPU
- The shape may deform in real-time

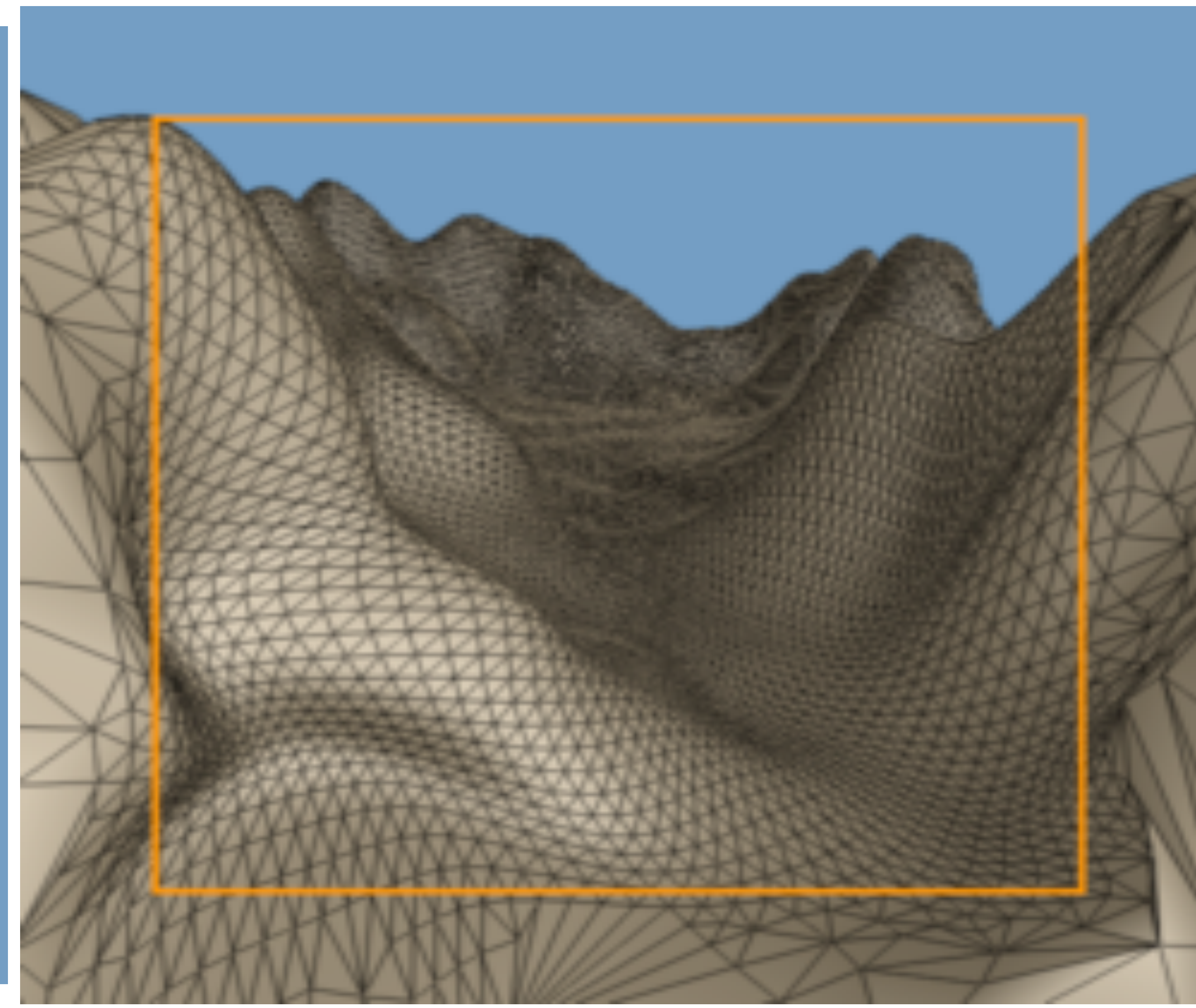


On-the-fly tessellation

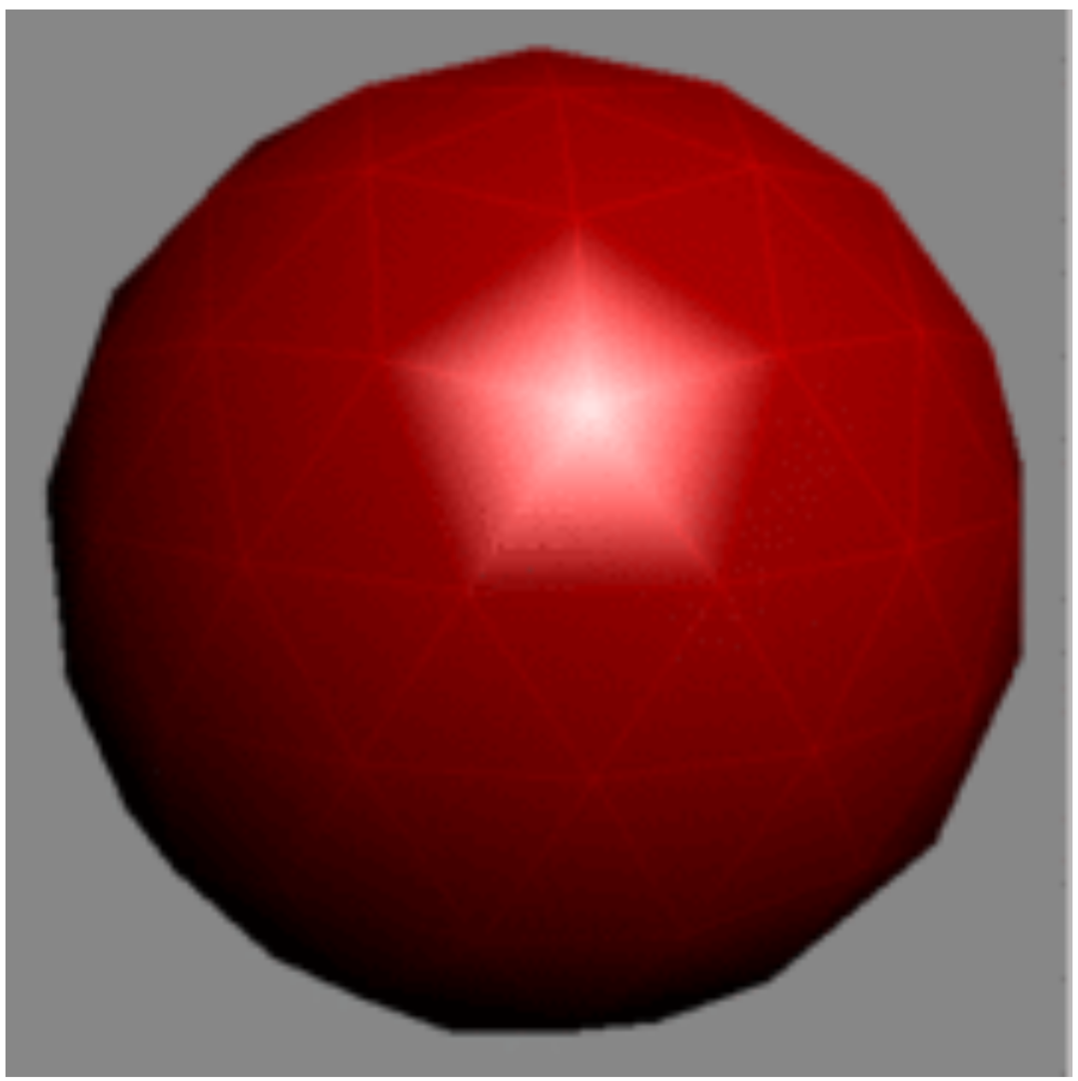
- So, say in a dynamic environment, what are the factors that we need to take into account when doing the tessellation?
  - in addition to curvature?

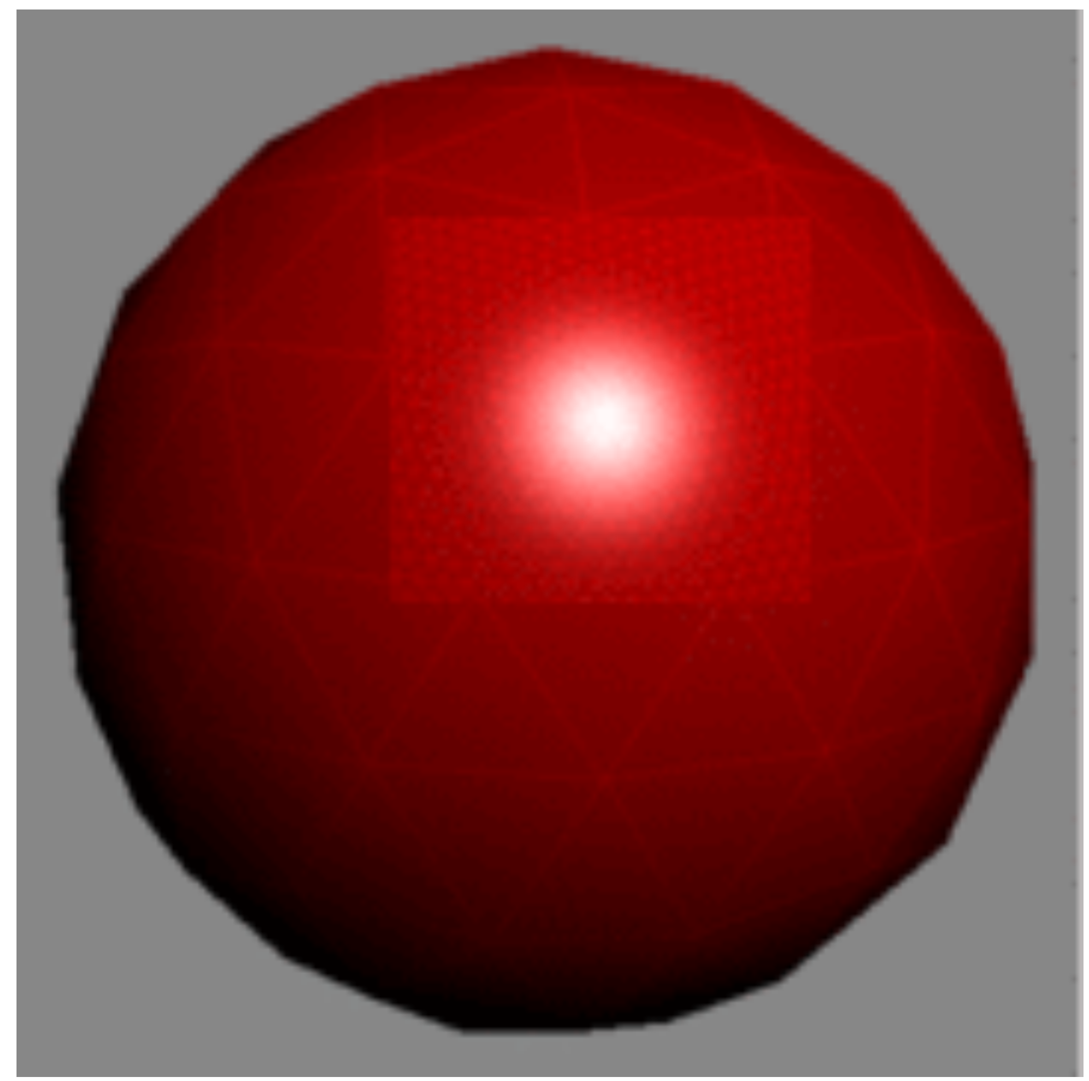
Other factors?



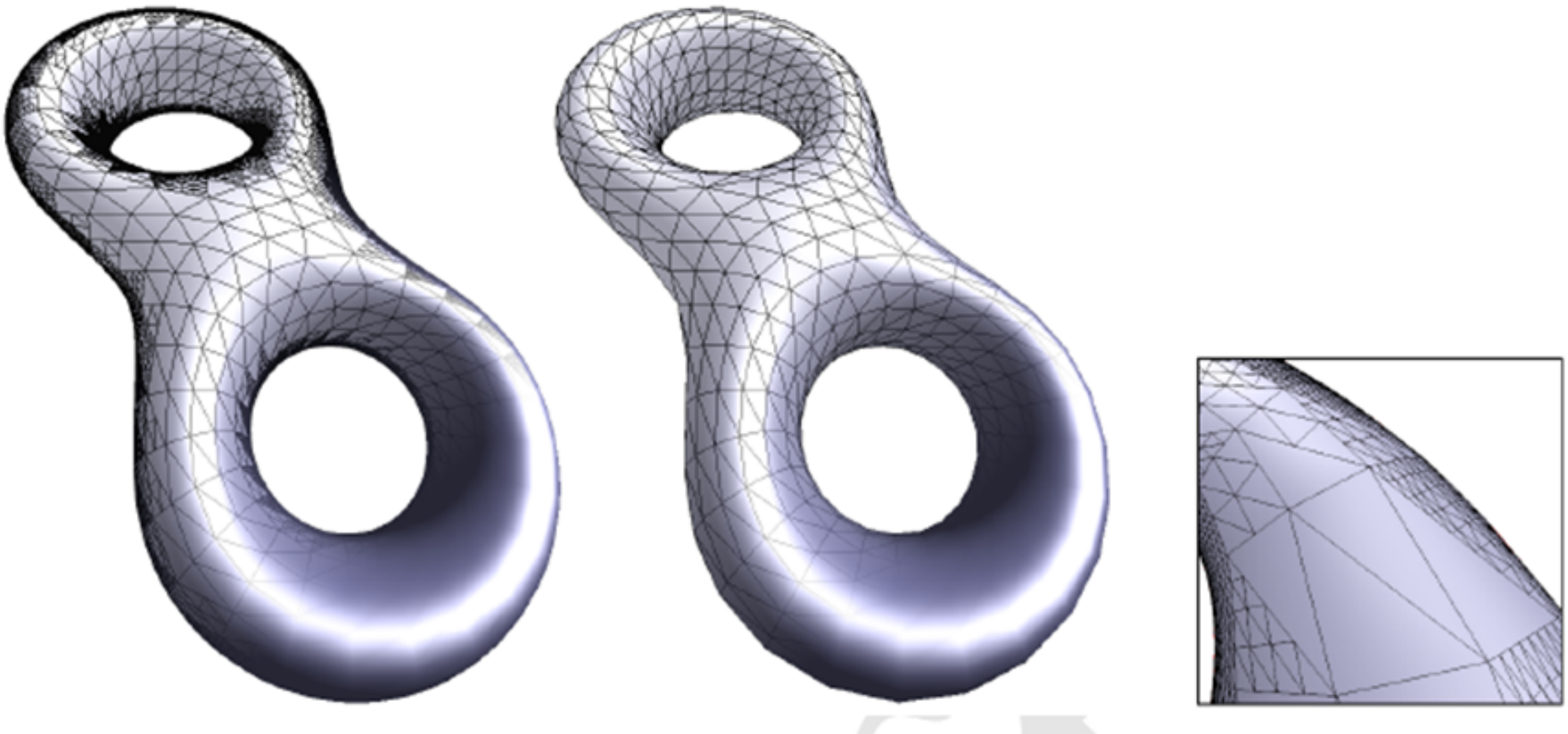


Other factors?





Other factors?



Other factors to evaluate

- Inside the view frustum
- Front facing
- Occupying a large area in screen space
- Close to the silhouette of the object
- Illuminated by a significant amount of specular lighting

Summary

- Hermite, Bezier, B-Spline curves
- Bicubic patches
- Tessellation
  - Triangulation of parametric surfaces
  - On-the-fly tessellation

References

- Shirley Chapter 15 (Curves)
- Foley et al. Chapter 11, section 11.2 up to and including 11.2.3
- Foley at al., Chapter 11, sections 11.2.9, 11.2.10, 11.3 and 11.5.
- Akenine-Möller 13.6

Links

- http://www.rose-hulman.edu/~finn/CCLI/Applets/CubicHermiteApplet.html
- http://www.rose-hulman.edu/~finn/CCLI/Applets/BezierBernsteinApplet.html
- http://www.rose-hulman.edu/~finn/CCLI/Applets/BSplineApplet.html
- http://www.personal.psu.edu/dpl14/java/parametricequations/beziersurfaces/index.html