Maths for computer graphics

Computer graphics involves lots of maths

- 2D and 3D coordinates
- Matrix transformations
- Reflection and refraction
- Curved surfaces

Vector spaces

Laws

$$
\begin{aligned}
\boldsymbol{a} + \boldsymbol{b} &= \boldsymbol{b} + \boldsymbol{a} \\
(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} &= \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \\
\lambda(\boldsymbol{a} + \boldsymbol{b}) &= \lambda\boldsymbol{a} + \lambda\boldsymbol{b} \\
(\lambda\mu)\boldsymbol{a} &= \lambda(\mu\boldsymbol{a}) \\
\boldsymbol{a} + \boldsymbol{0} &= \boldsymbol{a} \\
\forall\boldsymbol{a}\exists\boldsymbol{b} : \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{0} & \\
1\boldsymbol{a} &= \boldsymbol{a}
\end{aligned}
$$

E.g. two dimensions

$$
\begin{aligned}
\boldsymbol{u} &= \begin{pmatrix} x \\ y \end{pmatrix} \\
\|\boldsymbol{u}\| &= \sqrt{x^2 + y^2}
\end{aligned}
$$

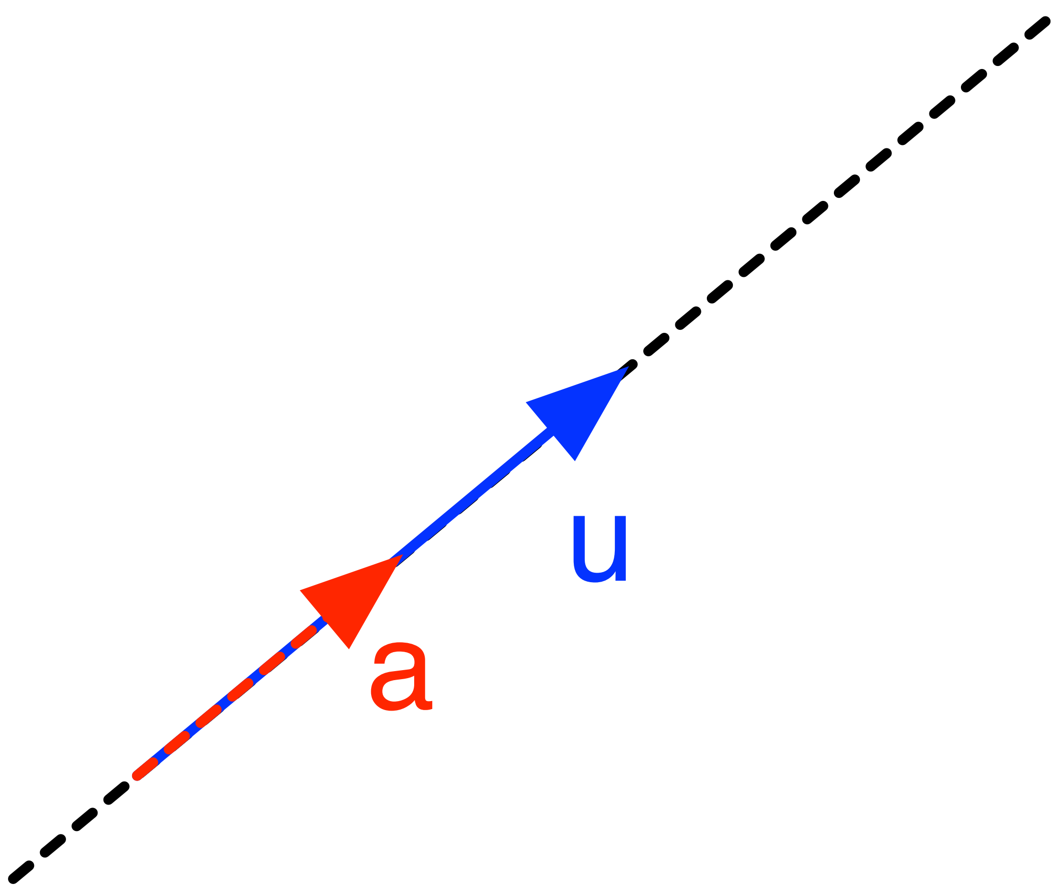
N dimensions

$$
\begin{aligned}
\boldsymbol{v} &= \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \\
\|\boldsymbol{v}\| &= \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}
\end{aligned}
$$

Subspaces

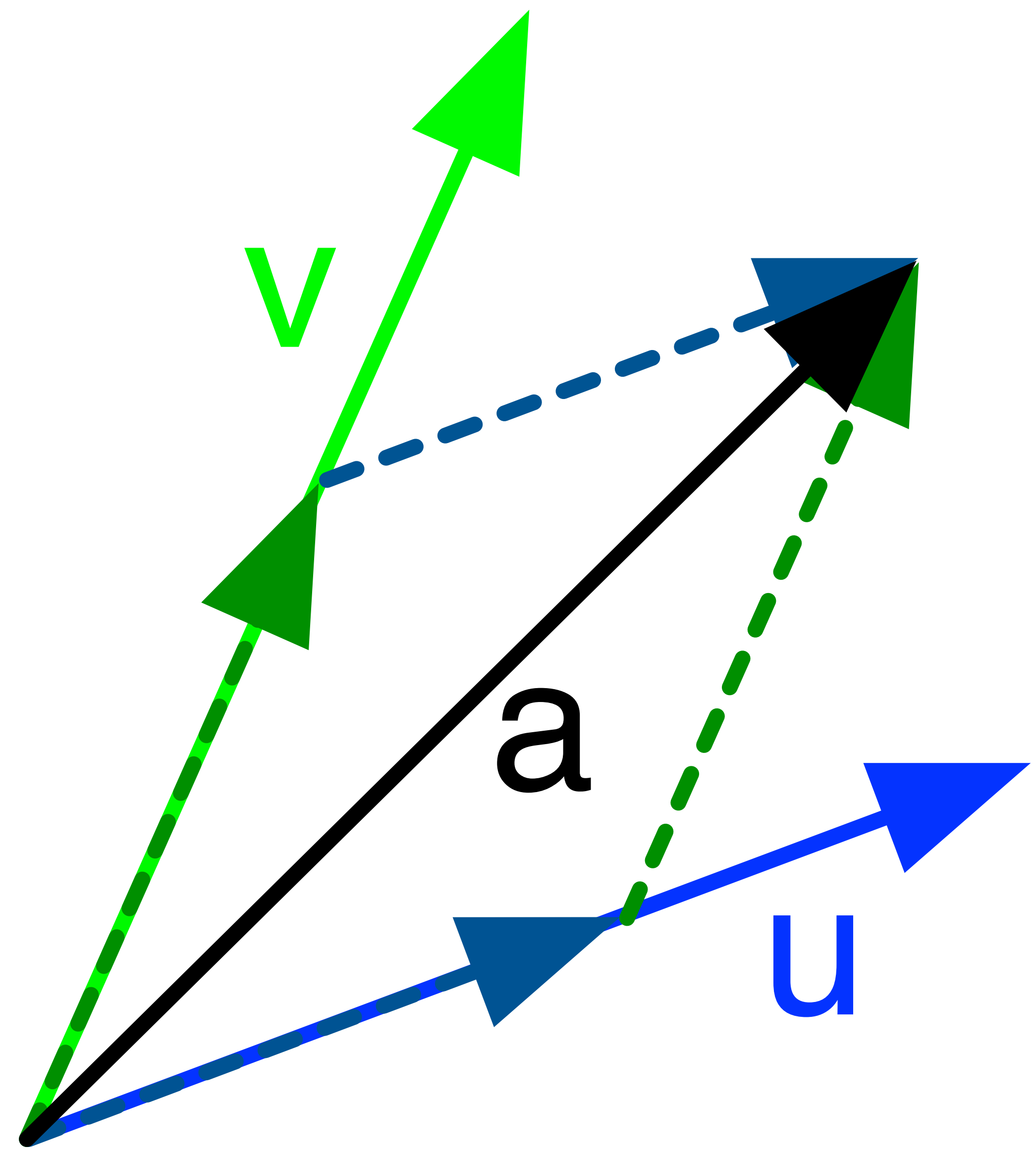
Line

$\boldsymbol{a} = \lambda \boldsymbol{u}$



Plane

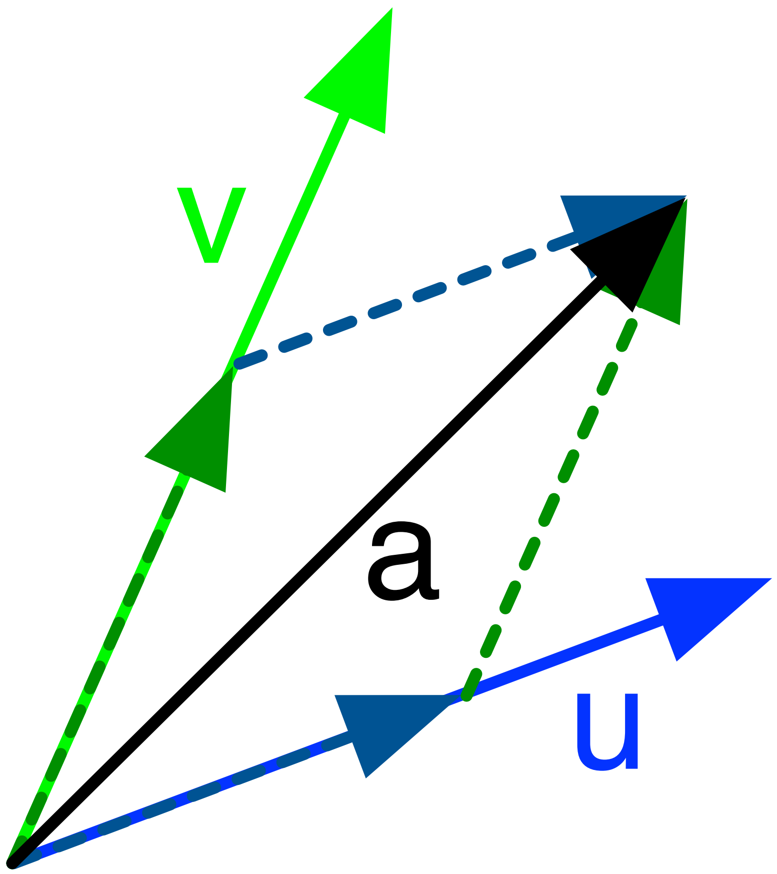
$\boldsymbol{a} = \lambda \boldsymbol{u} + \mu \boldsymbol{v}$



3D space

$\boldsymbol{a} = \lambda \boldsymbol{u} + \mu \boldsymbol{v} + \nu \boldsymbol{w}$

Linear dependence



Two vectors are *linearly dependent* if:

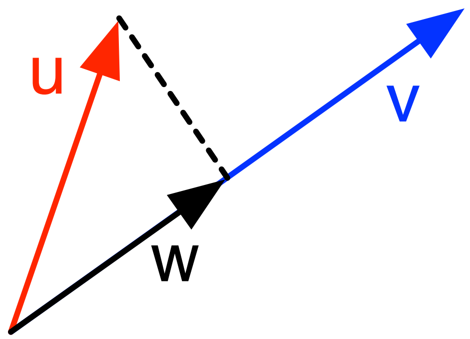
- $\alpha \boldsymbol{a} + \beta \boldsymbol{b} = \boldsymbol{0}$

for some α and β not both zero. More generally for *n* vectors:

- $\alpha \boldsymbol{a} + \beta \boldsymbol{b} + \ldots + \nu \boldsymbol{n} = \boldsymbol{0}$

Dot product

$$\boldsymbol{u} \cdot \boldsymbol{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$



$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos\theta \\ \boldsymbol{u} \cdot \boldsymbol{v} &= \boldsymbol{v} \cdot \boldsymbol{u} \\ \boldsymbol{u} \cdot (\lambda \boldsymbol{v} + \mu \boldsymbol{w}) &= \lambda(\boldsymbol{u} \cdot \boldsymbol{v}) + \mu(\boldsymbol{u} \cdot \boldsymbol{w}) \end{aligned}$$

$$\|w\| = u \cdot v$$

Coordinate systems

In an N dimensional vector space V, a set of N linearly independent vectors forms a *basis*. For some vector $\boldsymbol{u} \in V$:

$$\boldsymbol{u} = \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_2 + \ldots + \nu \boldsymbol{e}_N$$

A basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ is *orthogonal* if for all $\boldsymbol{e}_i, \boldsymbol{e}_j, i \neq j$:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$$

A basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ is *orthonormal* if it is orthogonal and for all $\boldsymbol{e}_i$:

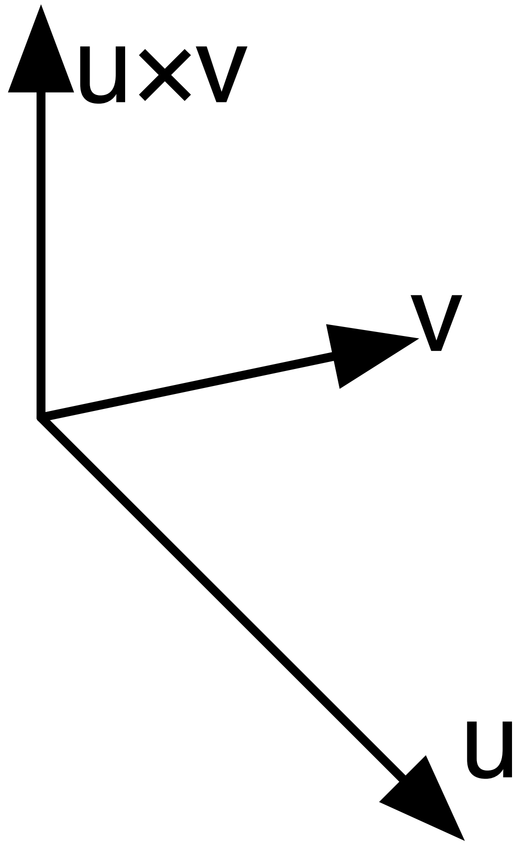
$$\|\boldsymbol{e}_i\| = 1$$

Cross product

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \sin\theta$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$$

$$\boldsymbol{v} \times \boldsymbol{u} = -\boldsymbol{u} \times \boldsymbol{v} \neq \boldsymbol{u} \times \boldsymbol{v}$$



Matrix algebra

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$\begin{aligned} A(BC) &= (AB)C \\ AB &\neq BA \\ (A+B)C &= AC + BC \\ AI &= IA = A \\ (AB)^T &= B^T A^T \\ A^{-1}A &= I \end{aligned}$$

References

Reading

- Shirley, Chapter 2.4 (Miscellaneous Math – Vectors)
- Foley, Appendix A.1-A.5 (Maths for Computer Graphics)