Implicit lines and barycentric coordinates

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Implicit equations for 2D lines

We can define a 2D line using the simple equation

$$y = mx + d,$$

but for the purposes of computer graphics this is inconvenient since for vertical lines $m = \infty$. A more general form is:

$$ax + by + c = 0.$$

For a line with endpoints (x_p, y_p) and (x_q, y_q) we can derive

$$y = \frac{(y_q - y_p)}{(x_q - x_p)}x + d \tag{1}$$
$$0 = (y_q - y_p)x - (x_q - x_p)y + c, \tag{2}$$

where the constant terms have been collected into c. This gives us an equation for a straightline of the form $f(x, y) = 0$. Then since we know (x_p, y_p) is on the line, we can find c, as we know $f(x_p, y_p) = 0$

$$(y_q - y_p)x_p - (x_q - x_p)y_p + c = 0 \tag{3}$$
$$c = (x_q - x_p)y_p - (y_q - y_p)x_p \tag{4}$$
$$= x_q y_p - y_q x_p, \tag{5}$$

so then

$$f(x, y) = (y_q - y_p)x - (x_q - x_p)y + x_q y_p - y_q x_p = 0. \tag{6}$$

To find the normal vector of the line we simply find the gradient of the implicit function $f(x, y)$:

$$\boldsymbol{n} = \nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

For our implicit form of the line $f(x, y) = ax+by+c = 0$ this is simply $\boldsymbol{n} = (a, b)$. The perpendicular vector from the line to a point is then some multiple of this normal vector, i.e. $k\boldsymbol{n}$, and the length of this vector is $d = k\|\boldsymbol{n}\| = k\sqrt{a^2 + b^2}$.

Taking a point $u = (x+ka, y+kb)$ a distance $k\sqrt{a^2 + b^2}$ from the point $p = (x, y)$ on the line (see figure 1),

$$\begin{aligned} f(x + ka, y + kb) &= a(x + ka) + b(y + kb) + c && (7) \\ &= ax + ka^2 + by + kb^2 + c && (8) \\ &= k(a^2 + b^2). && (9) \end{aligned}$$

So the signed distance d to the line from some point (x, y) is a multiple of $f(x, y)$,

$$d = \frac{f(x, y)}{\sqrt{a^2 + b^2}}.$$

This is called the signed distance because it is positive on one side of the line (the side pointed to by the normal vector of the line), and negative on the other.

For an implicit 2D line of the form $ax - by + c = 0$, the normal vector is $(a, -b)$, and so as long as $b > 0$, the normal vector will point downwards (in the negative y axis direction). Therefore for points (x, y) below the line $f(x, y)$ will be positive, and for points above the line $f(x, y)$ will be negative. In the midpoint algorithm, the implicit equation for the line takes the form

$$f(x, y) = (y_r - y_l)x - (x_r - x_l)y + c = 0,$$

with (x_l, y_l) and (x_r, y_r) being the left and right endpoints of the line respectively. Then because $x_r - x_l > 0$ (since x_r is to the right of x_l), the normal vector of the line will point downwards, and points below the line will have a positive value of $f(x, y)$.

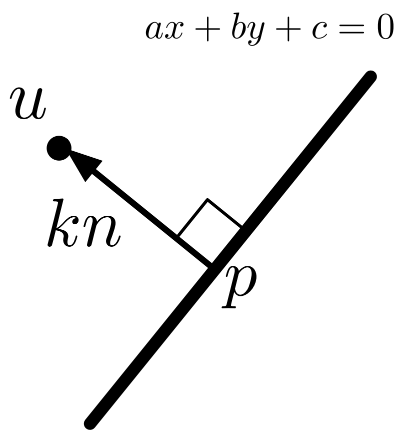


Figure 1: A point u a distance $k\|\boldsymbol{n}\|$ from the line. Here $\boldsymbol{n} = (a, b)$ since $f(x, y) = ax + by + c$.

Barycentric coordinates

Barycentric coordinates define points in a 2D plane based on the edges of a triangle. If we define $f_{pq}(x, y)$ as the implicit equation for the line between the points p and q with coordinates (x_p, y_p) and (x_q, y_q), given in equation 6, then for a point (x, y) we know that $f_{pq}(x, y)$ is a scaled perpendicular distance to the point (x, y) from the line between points p and q.

The barycentric coordinates of a point in the 2D plane are defined as the ratios shown in figure 2. For β, the barycentric coordinate is the perpendicular distance of the point (x, y) from the line between P_0 and P_2, scaled so that at P_1, $\beta = 1$.

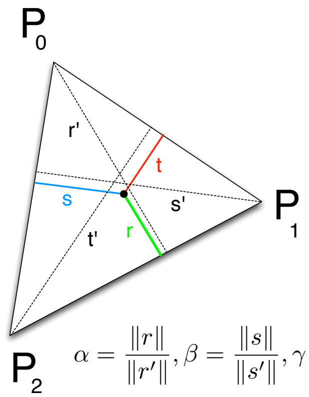
Then for a point (x, y) we can write

$$\beta = \frac{f_{P_2P_0}(x, y)}{f_{P_2P_0}(x_1, y_1)},$$

where $P_1 = (x_1, y_1)$. Similarly with $P_0 = (x_0, y_0)$ and $P_2 = (x_2, y_2)$ we can write

$$\gamma = \frac{f_{P_0P_1}(x, y)}{f_{P_0P_1}(x_2, y_2)},$$

and



$$\alpha = \frac{\|r\|}{\|r'\|}, \beta = \frac{\|s\|}{\|s'\|}, \gamma = \frac{\|t\|}{\|t'\|}$$

Figure 2: Definition of barycentric coordinates of a point. The vectors r, r', s, s', t and t' are perpendicular to the corresponding edges of the triangle (*e.g.* r and r' are perpendicular to the line between P_1 and P_2).

$$\alpha = \frac{f_{P_1 P_2}(x, y)}{f_{P_1 P_2}(x_0, y_0)}.$$

In words, γ is the perpendicular distance of the point (x, y) from the line between P_0 and P_1, scaled so that at P_2, $\gamma = 1$, and α is the perpendicular distance of the point (x, y) from the line between P_1 and P_2, scaled so that at P_0, $\alpha = 1$.

The division of *e.g.* $f_{P_2 P_0}(x, y)$ by $f_{P_2 P_0}(x_1, y_1)$ is simply to scale β so that at P_1, where $x = x_1$ and $y = y_1$, $\beta = \frac{f_{P_2 P_0}(x_1, y_1)}{f_{P_2 P_0}(x_1, y_1)} = 1$