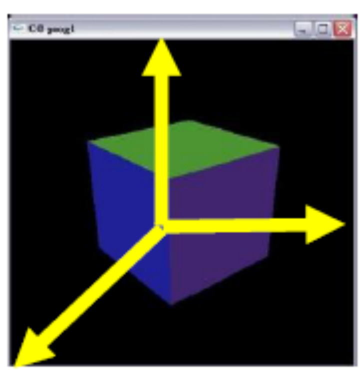
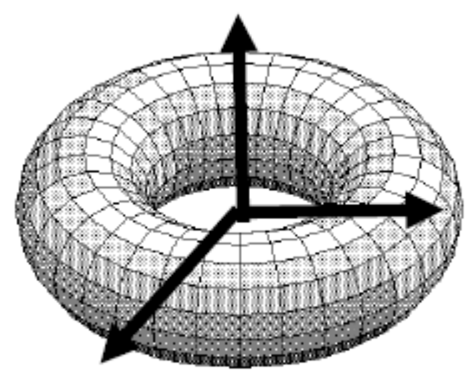
Transformations

Computer Graphics
Taku Komura

Setting Objects in the Scene

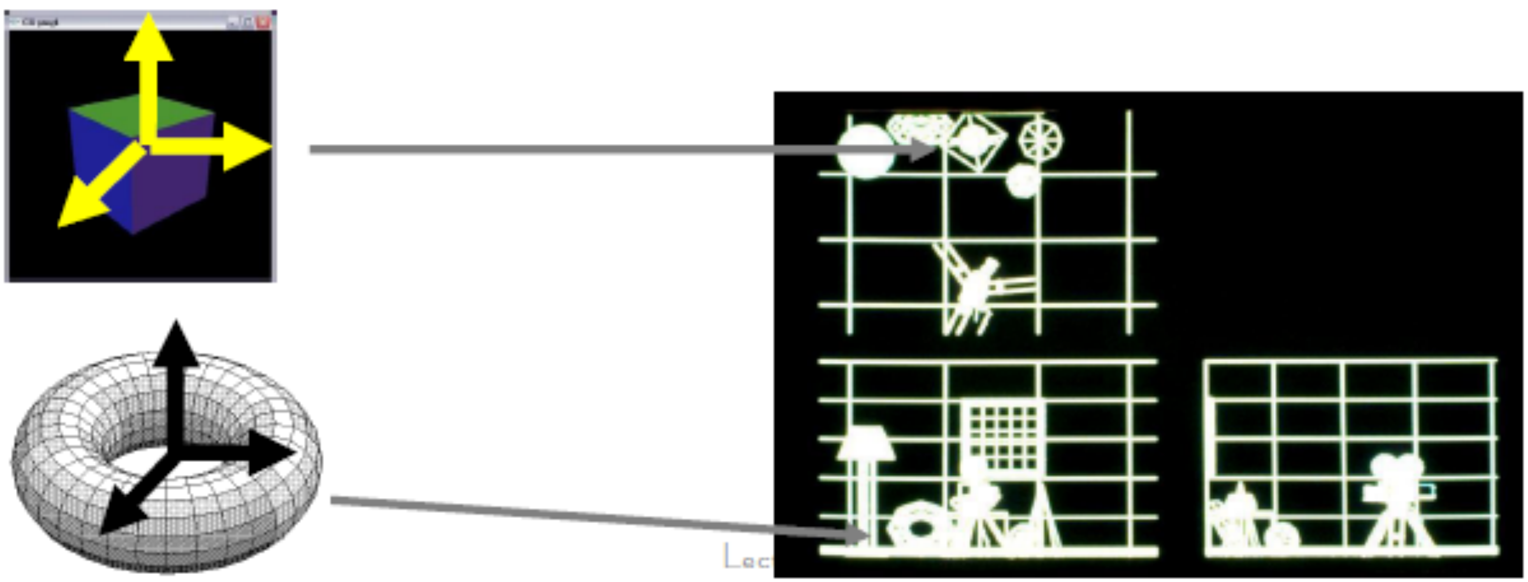
- Once the models are prepared, we need to place them in the environment
- We need to know the vertex locations of the objects in the world coordinate system
- But objects are only defined in their own local coordinate system





Transformations

We translate, rotate and scale the vertices in the world coordinate system



Overview

- Homogeneous transformation
  - Background
  - Conversion of coordinates
  - 2D, 3D cases
- Camera transformation

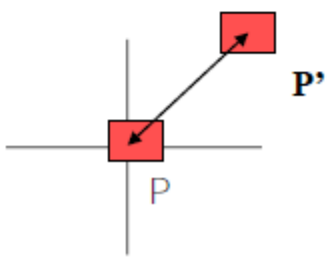
Homogenous Transformation

- Background
- Conversion of coordinates
- 2D, 3D cases

Background of Homogeneous Transformation

Translation

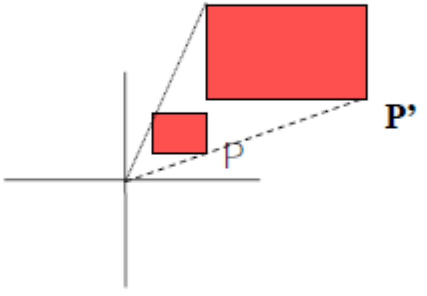
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$



Background of Homogeneous

Scaling

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

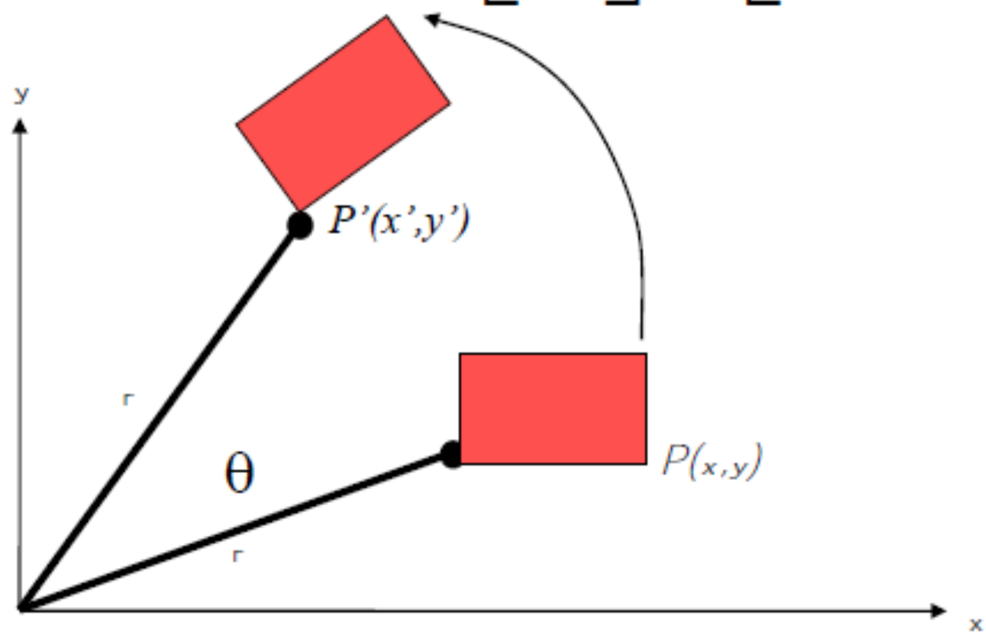


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

Background of Homogeneous

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



Background of Homogeneous

Basic transformations

- Translation.
  - $P' = T + P$
- Scale
  - $P' = S \cdot P$
- Rotation
  - $P' = R \cdot P$
- Different operations for translation
- We prefer a consistent operation for all three transformation
- $\Rightarrow$ homogenous transformation.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous Transformation

- Translation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + d_x \\ y + d_y \\ 1 \end{pmatrix}$$

- Rotation

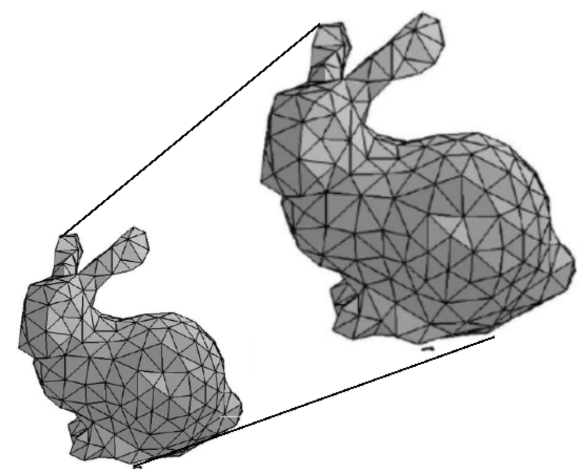
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Scaling

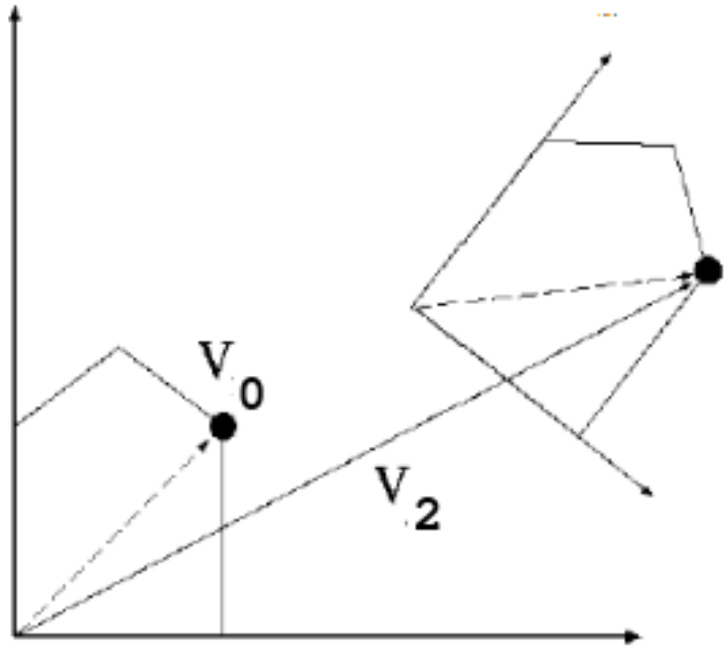
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Subsequently applying transformations?

- We can apply the above mentioned transformations to translate, scale and rotate the object vertices
- What if we want a combination of such operations?



Subsequently applying transformations



- $v_1 = M_0 v_0, \quad v_2 = M_1 v_1$

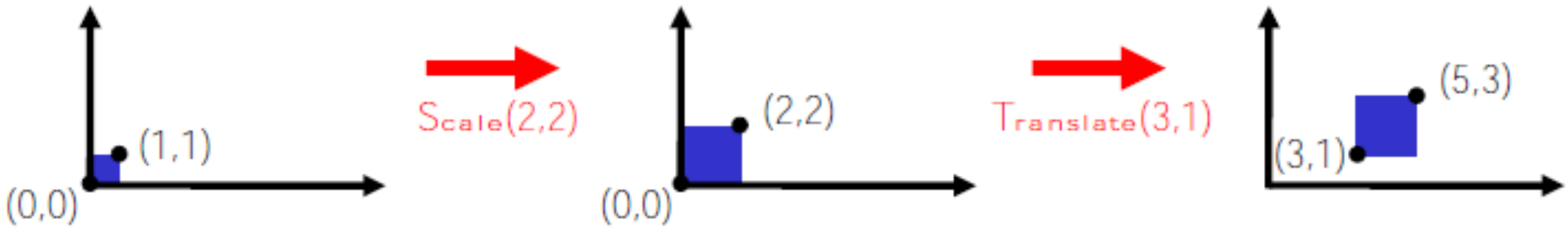
$$\longrightarrow \quad v_2 = M_1 M_0 v_0 = M_{1,0} v_0$$

- Can sequentially multiply matrices

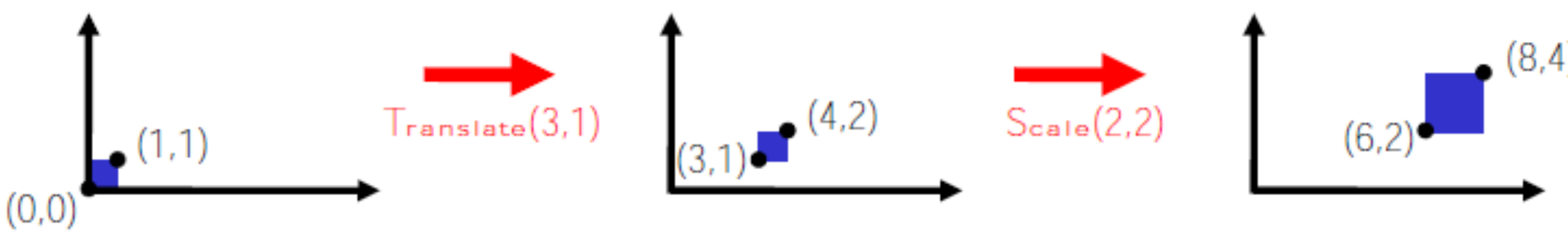
Subsequently applying transformations (2)

- Matrix operations are not commutative
- $AB \neq BA$

Scale then Translate: $p' = T(Sp) = TSp$



Translate then Scale: $p' = S(Tp) = STp$



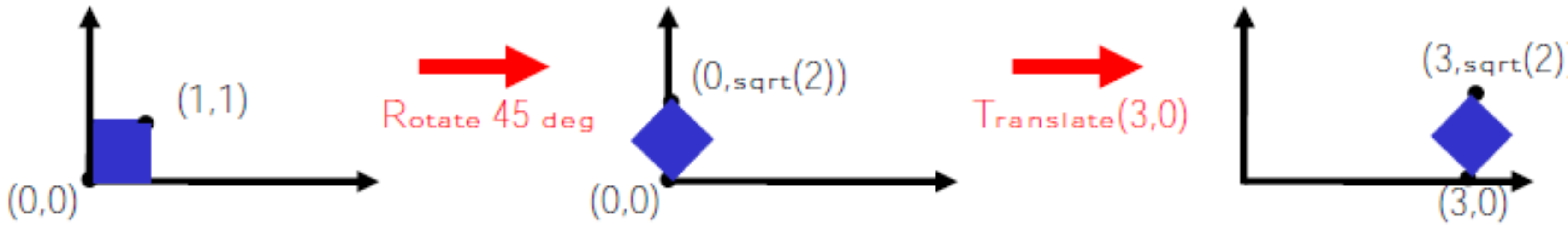
Scale then Translate: $p' = T(Sp) = TSp$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

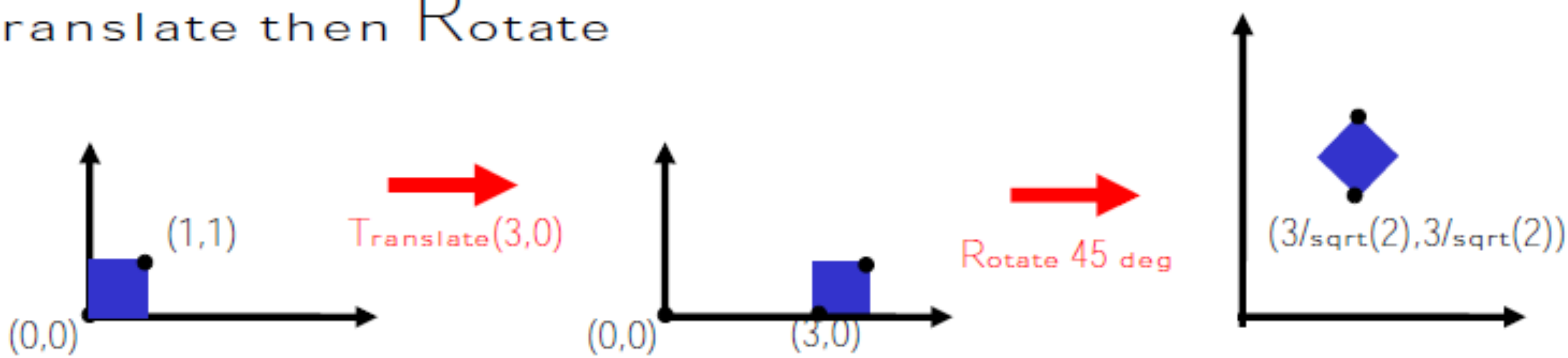
Translate then Scale: $p' = S(Tp) = STp$

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotate then Translate



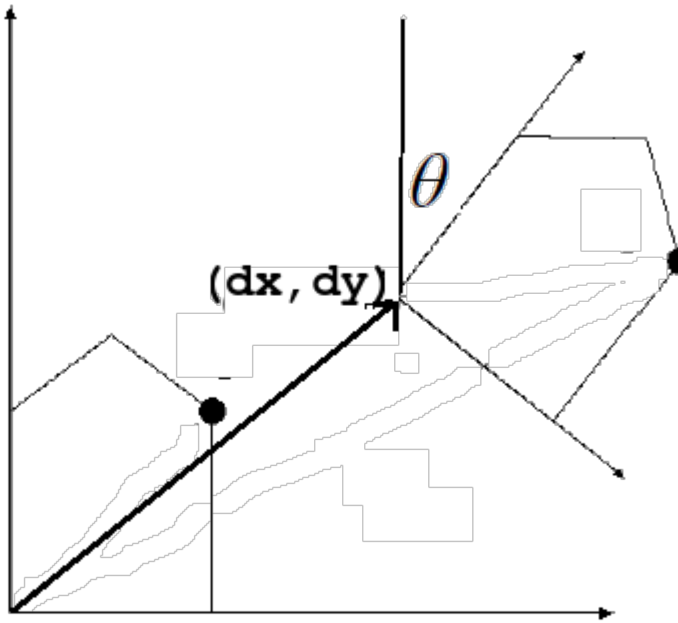
Translate then Rotate



Rotation, Translation

The matrix of rotation and then translation is often used so it may be worth to remember it.

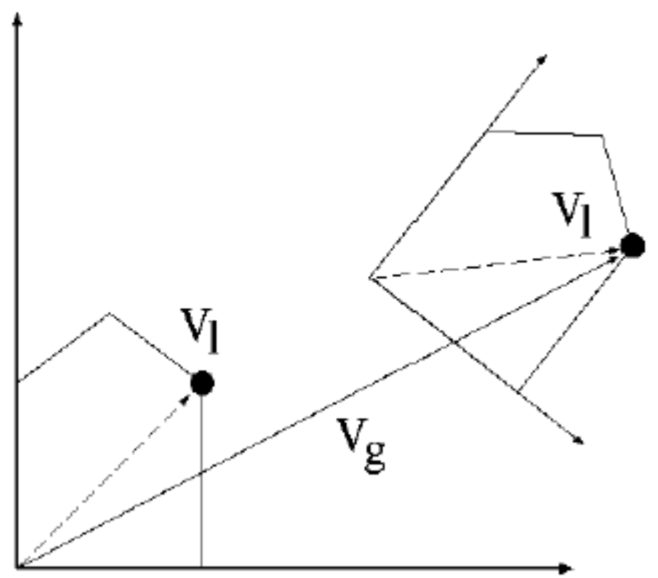
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Transformations between different coordinate systems

We can interpret that the transformation matrix is converting the location of vertices between different coordinate systems

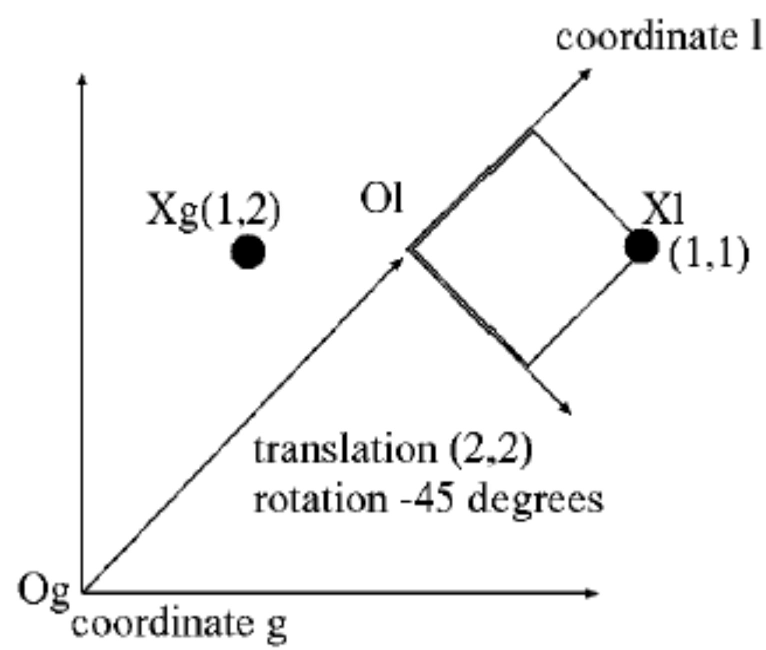
- $v_g = M v_l$
- $v_l = M^{-1} v^g$



Example

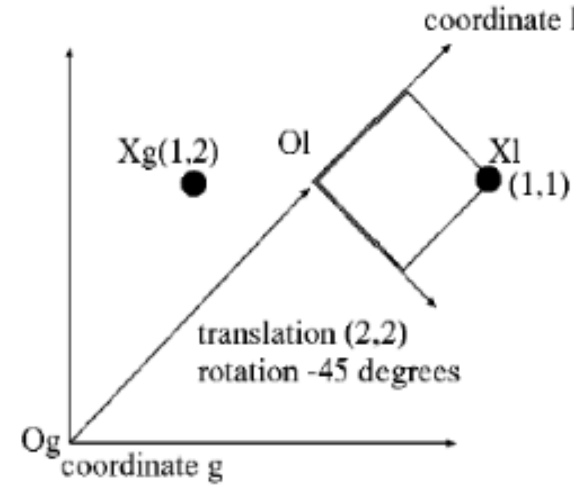
- What is the position of Xg in coordinate system l?
- What is the position of Xl in coordinate system g?
- Matrix for transformation between the two are

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}.$$



$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+\sqrt{2} \\ 2 \\ 1 \end{bmatrix}$$

Xl viewed from g



$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

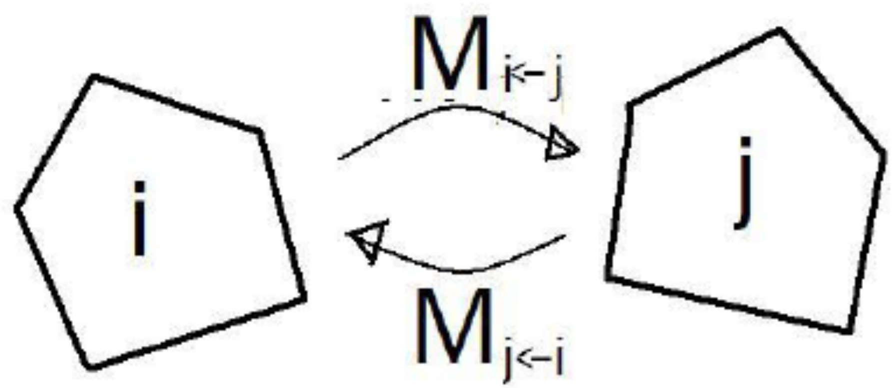
Xg viewed from l

Transformations between different coordinate systems (2)

We can forget about local-global and generalize the idea

$$v_i = M_{i \leftarrow j} v_j$$

$$v_j = M_{j \leftarrow i} v_i = M_{i \leftarrow j}^{-1} v_i$$



Sequential Transformations between different coordinate systems

We have a series of coordinate systems, a,b,c,d.

- We only know the relative transformations between the adjacent ones:

$$M_{a \leftarrow b} M_{b \leftarrow c} M_{c \leftarrow d}$$

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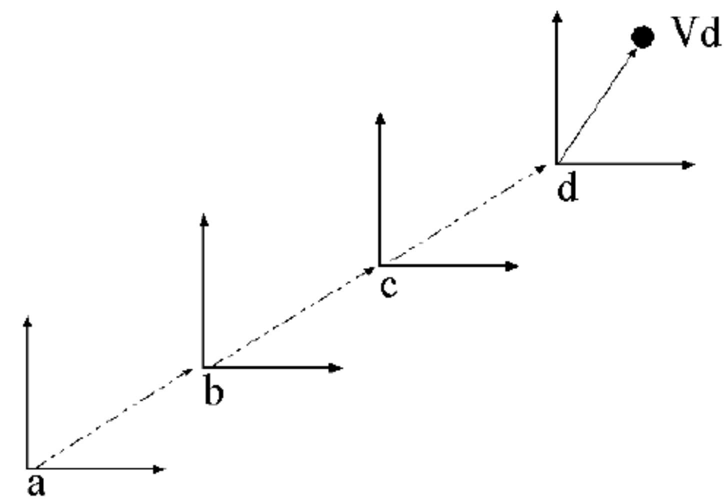
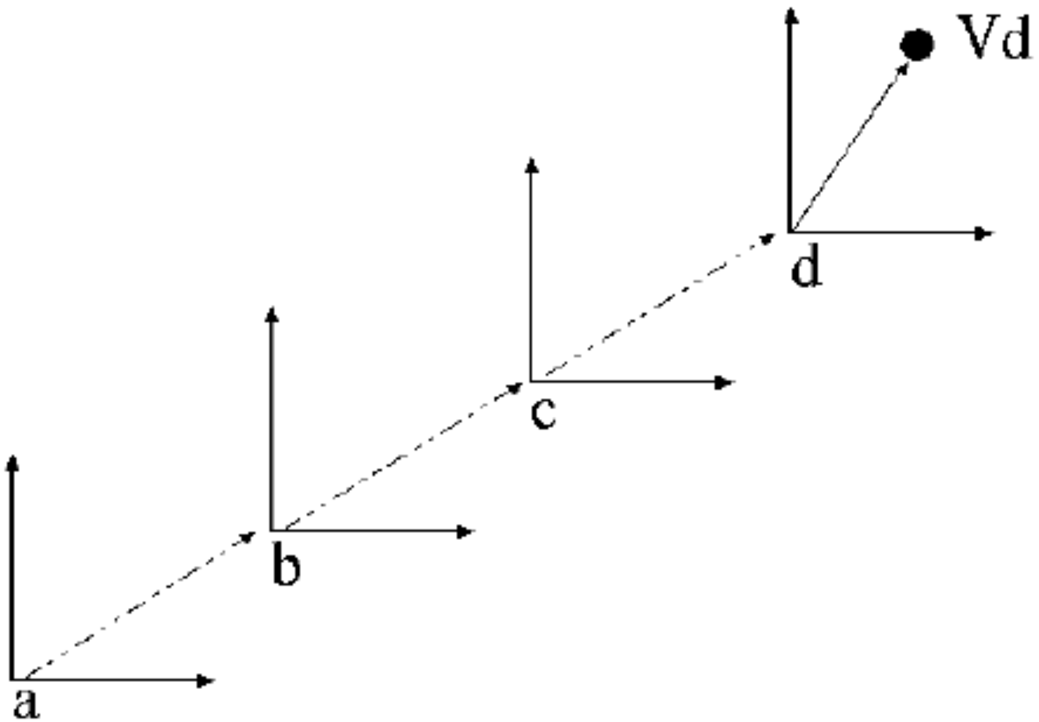
- What is the position of a point vd defined in coordinate d, when viewed from coordinate a?

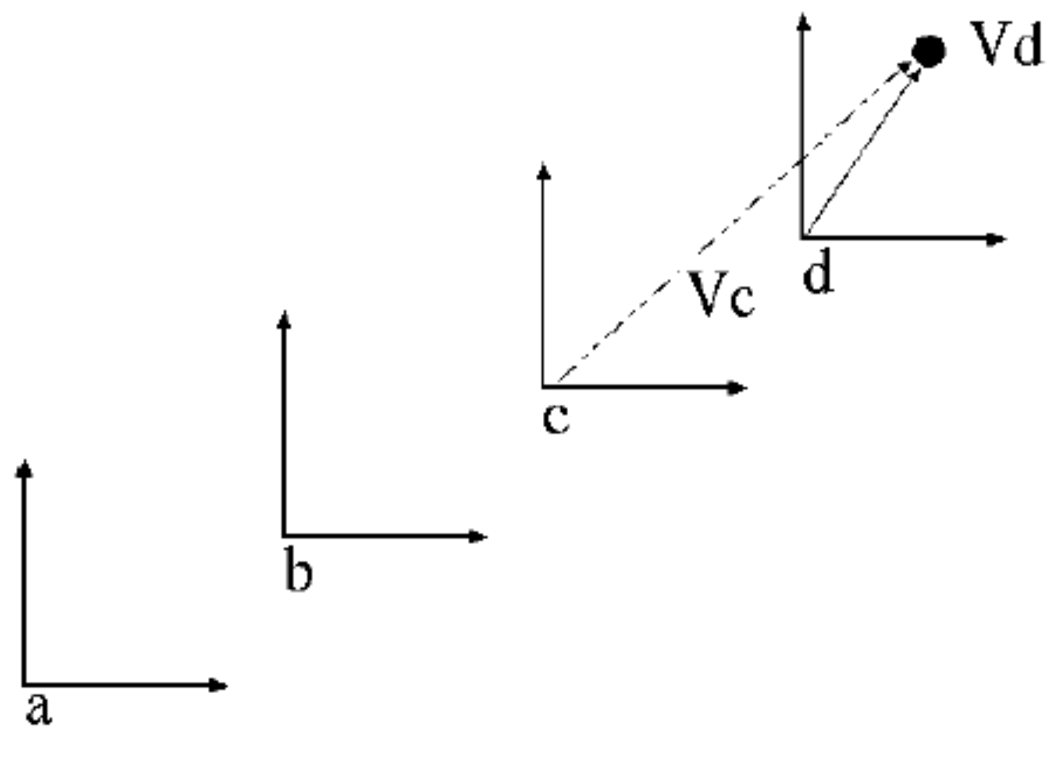


What is the position of Vd with respect to c?

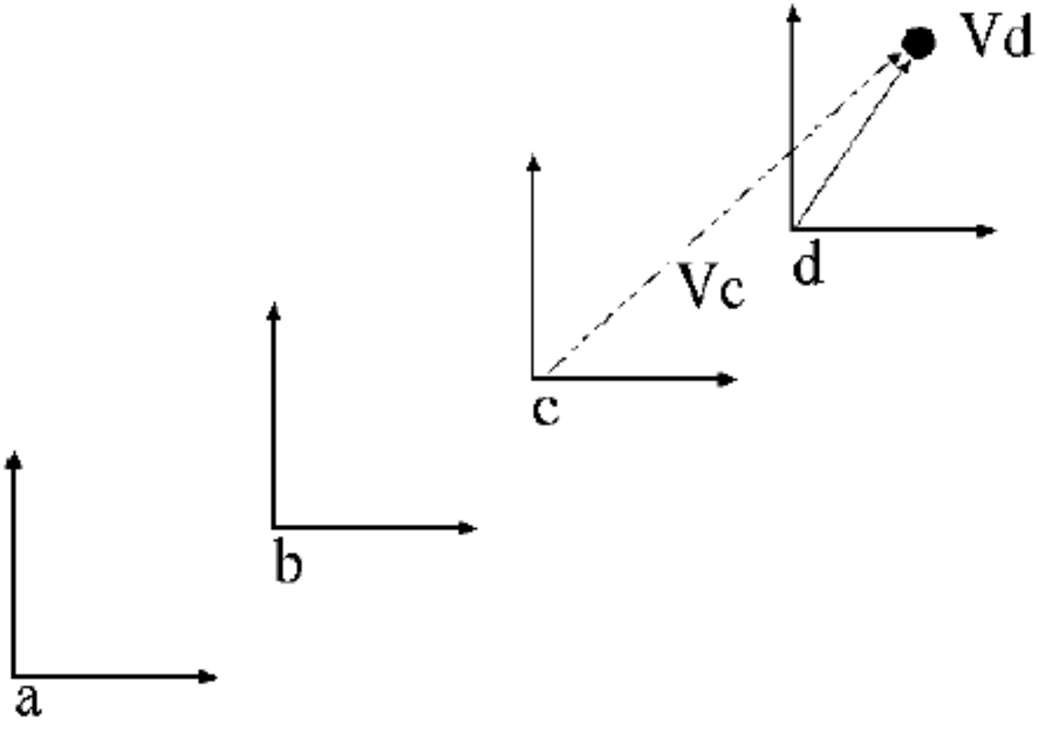


What is the position of Vd with respect to c?

- $v_c = M_{c \leftarrow d} v_d$

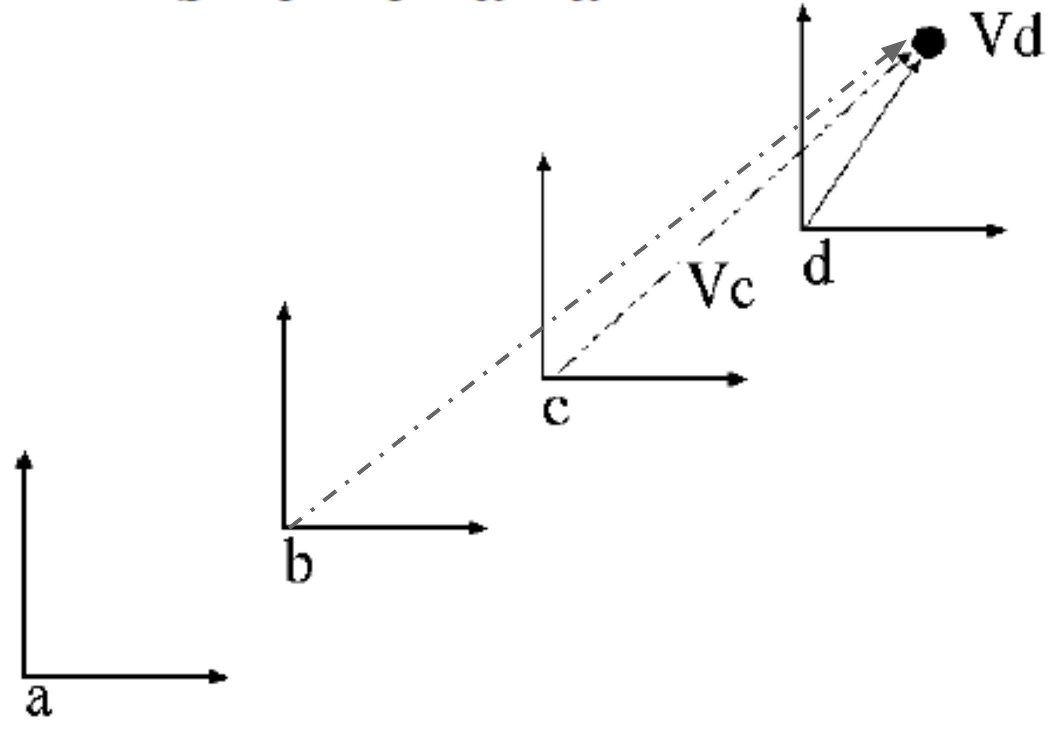


What is the position of Vd with respect to b?



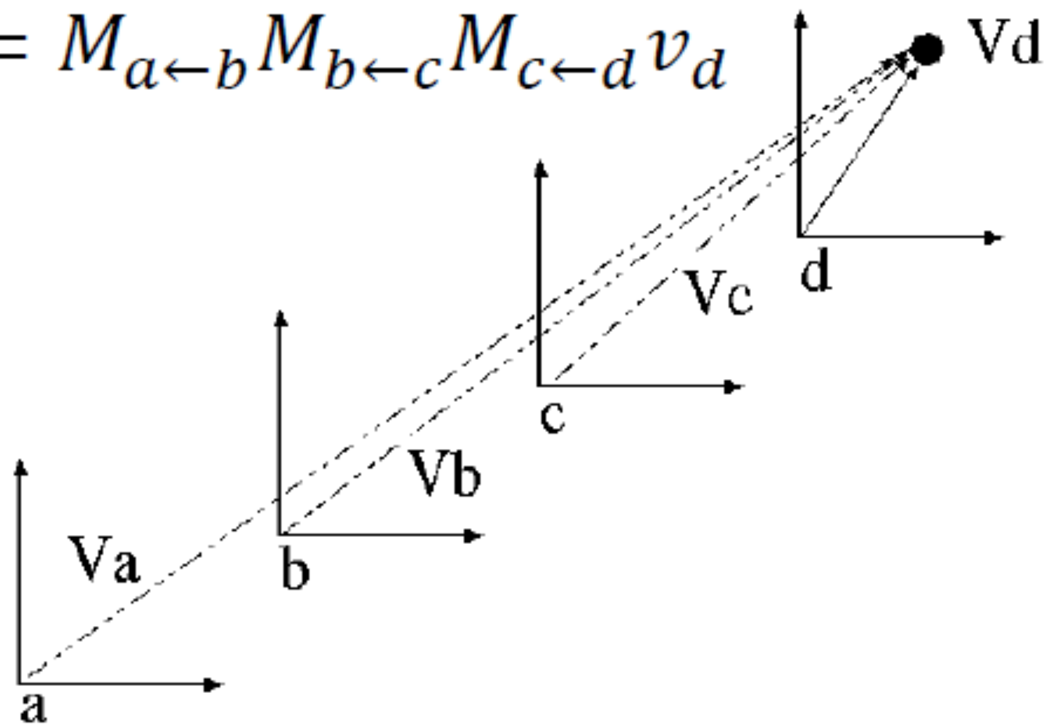
What is the position of Vd with respect to b?

- $v_c = M_{c \leftarrow d} v_d$
- $v_b = M_{b \leftarrow c} v_c = M_{b \leftarrow c} M_{c \leftarrow d} v_d$

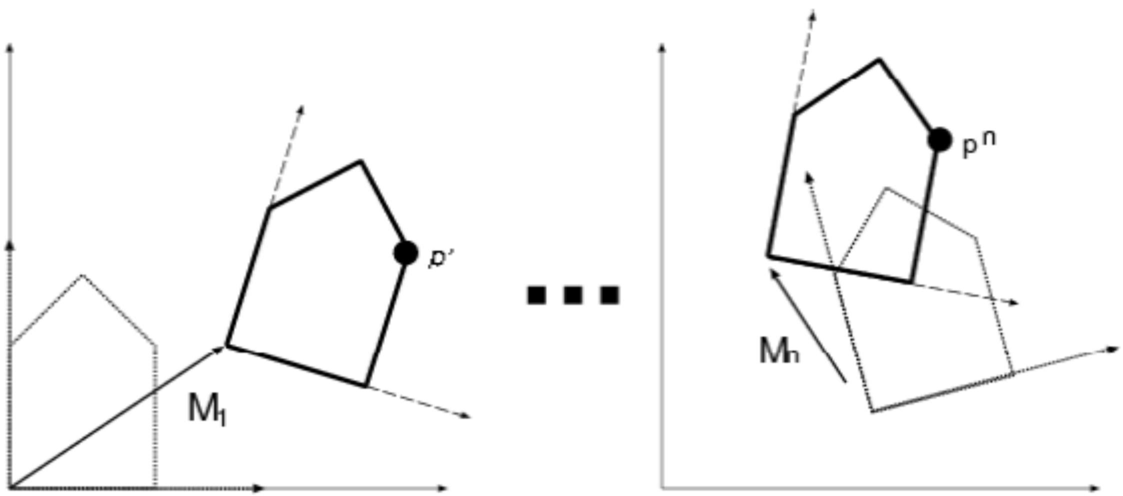


What is the position of Vd with respect to a?

- $v_c = M_{c \leftarrow d} v_d$
- $v_b = M_{b \leftarrow c} v_c = M_{b \leftarrow c} M_{c \leftarrow d} v_d$
- $v_a = M_{a \leftarrow b} v_b = M_{a \leftarrow b} M_{b \leftarrow c} M_{c \leftarrow d} v_d$



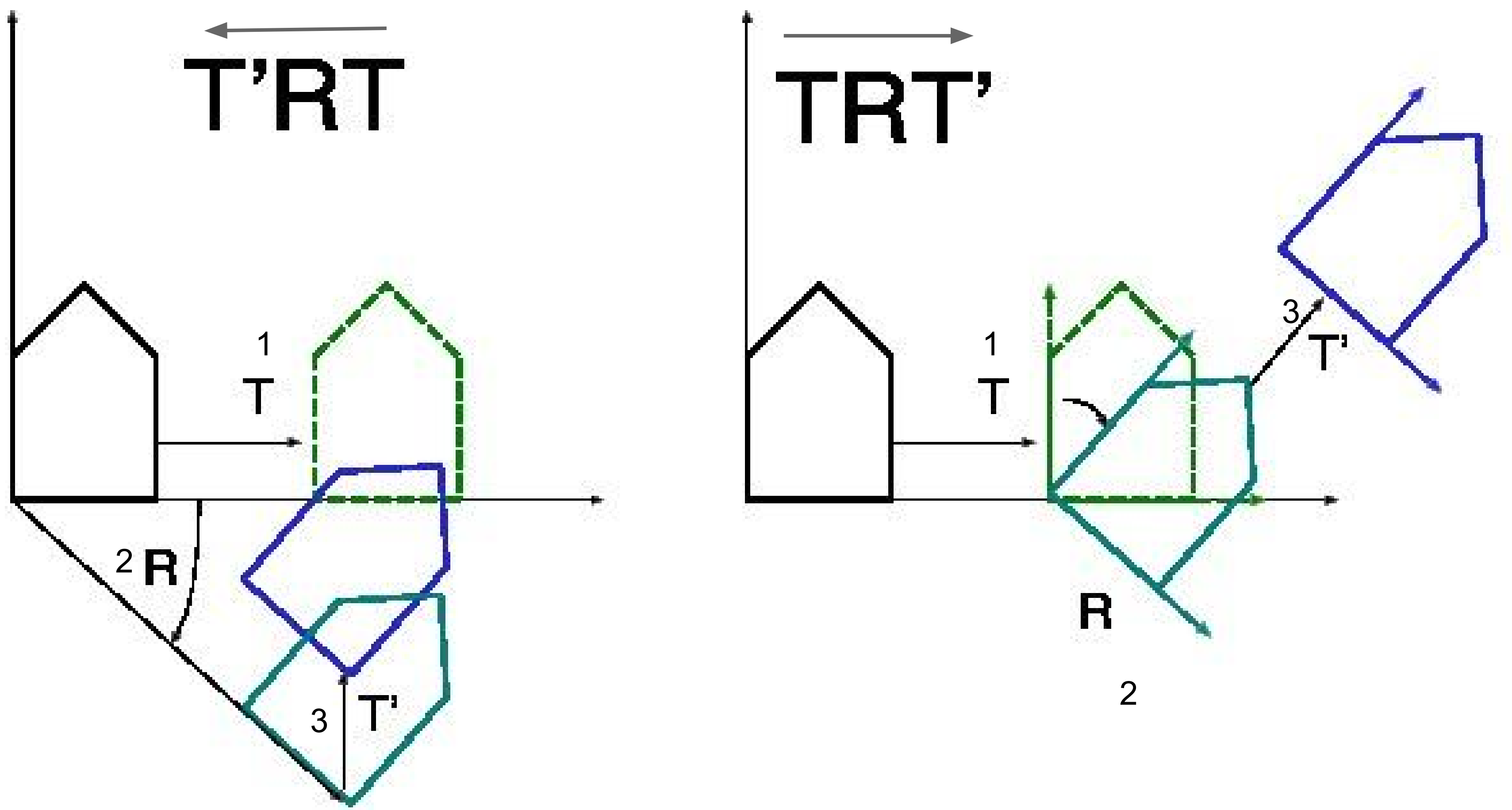
Transformations of coordinate



$$p^n = M_1 M_2 \cdots M_n p$$

A series of transformation matrices multiplied from left to right

Multiplying from the left and right

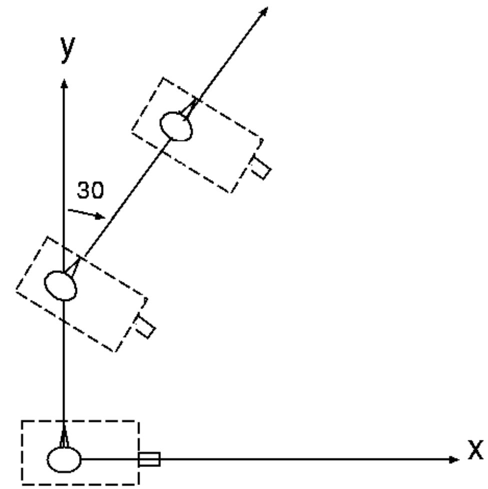


Why is this useful?

Car example:

–Starting the car from one location, we do a series of operations (driving forward, turning)

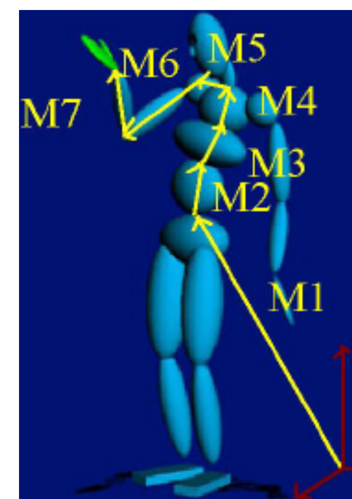
- What is the final location of the car?



Robot Example

- The body size of the robots are known and the robot bends it joints

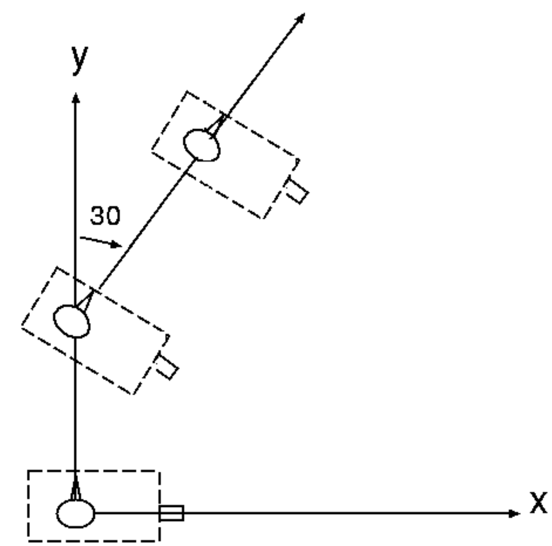
- What is the position of its hand?



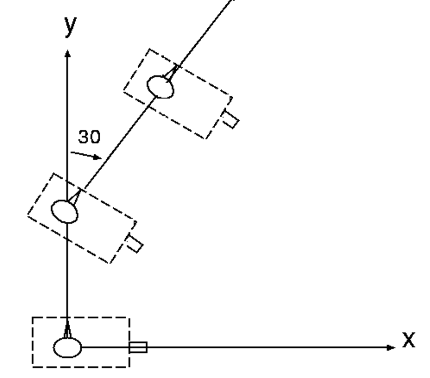
Quiz

I sat in the car, and find the side mirror is 0.4m on my right

- I started my car and drove 5m forward, turned 30 degrees to right, moved 5m forward again
- What is the position of the side mirror now, relative to where I was sitting in the beginning?



Hint



The matrix of first driving forward 5m is

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix to turn to the right 30degrees (rotating -30 around the origin) is

$$R_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

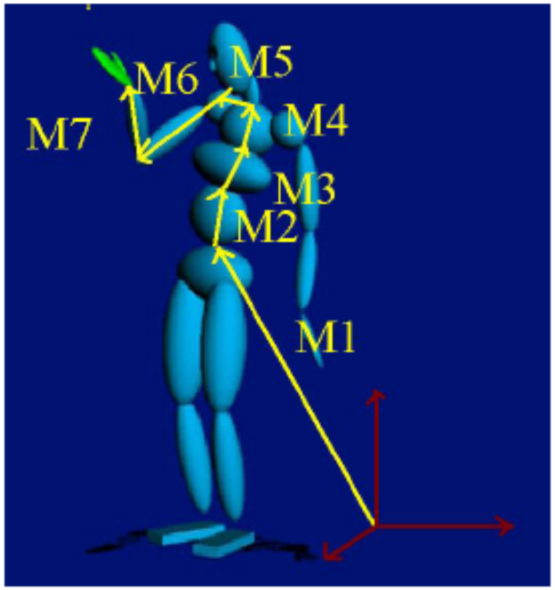
Solution

The local-to-global transformation matrix at the last configuration of the car is

$$M\mathbf{x} = TR_1T\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0 \\ 1 \end{bmatrix}$$

- A robot arm control: We know the position of the basis, and the joint angles at each joint. What is the position of the end effector?

$$M_1 M_2 M_3 M_4 M_5 M_6 M_7 x$$



Translation in 3D.

Simple extension to the 3D case:

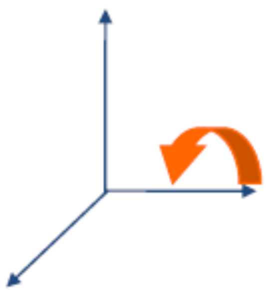
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Scale in 3D.

Simple extension to the 3D case:

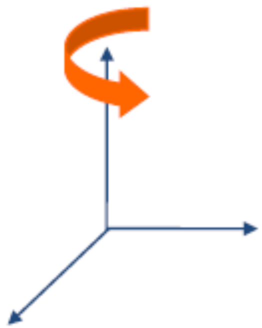
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotating About the x-axis $R_x(\theta)$



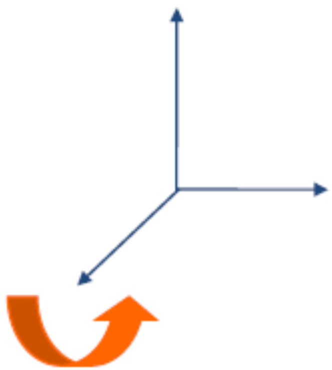
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotating About the y-axis
$R_y(\theta)$



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

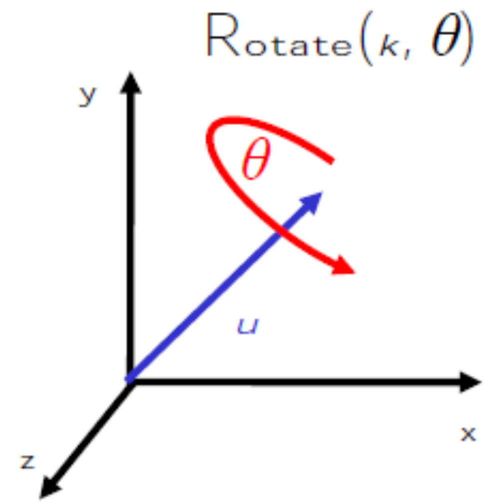
Rotation About the z-axis
$R_z(\theta)$



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation about an arbitrary axis

- About (u_x, u_y, u_z), a unit vector on an arbitrary axis

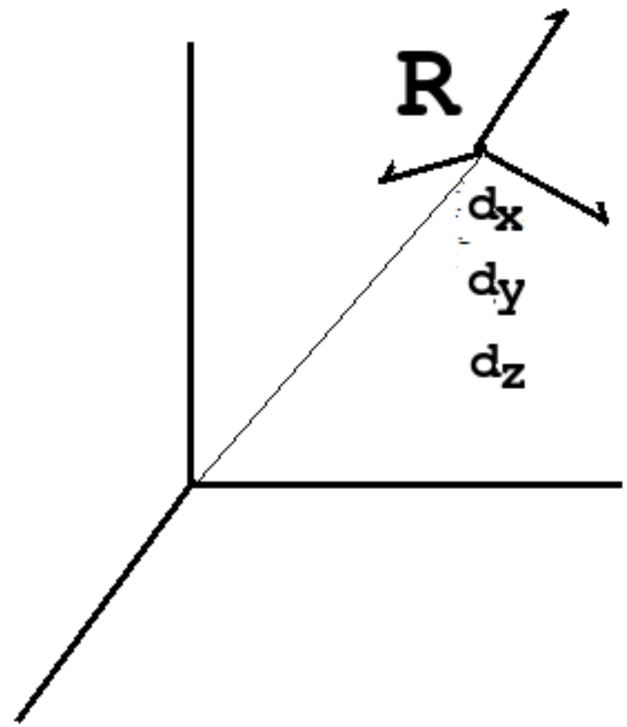


$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} u_x u_x(1-c)+c & u_z u_x(1-c)-u_z s & u_x u_z(1-c)+u_y s & 0 \\ u_y u_x(1-c)+u_z s & u_z u_x(1-c)+c & u_y u_z(1-c)-u_x s & 0 \\ u_z u_x(1-c)-u_y s & u_y u_z(1-c)+u_x s & u_z u_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where $c = \cos\theta$ & $s = \sin\theta$

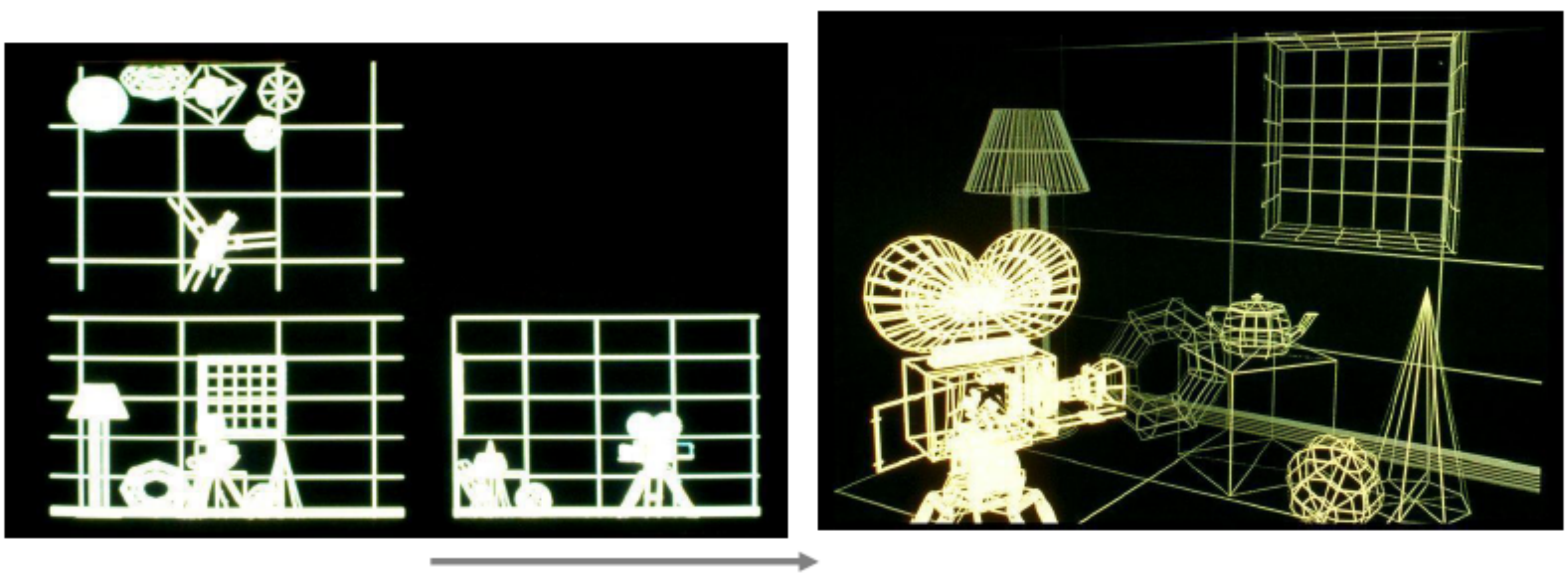
Rotation and Translation in 3D

$$\begin{pmatrix} & & & d_x \\ & R & & d_y \\ & & & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



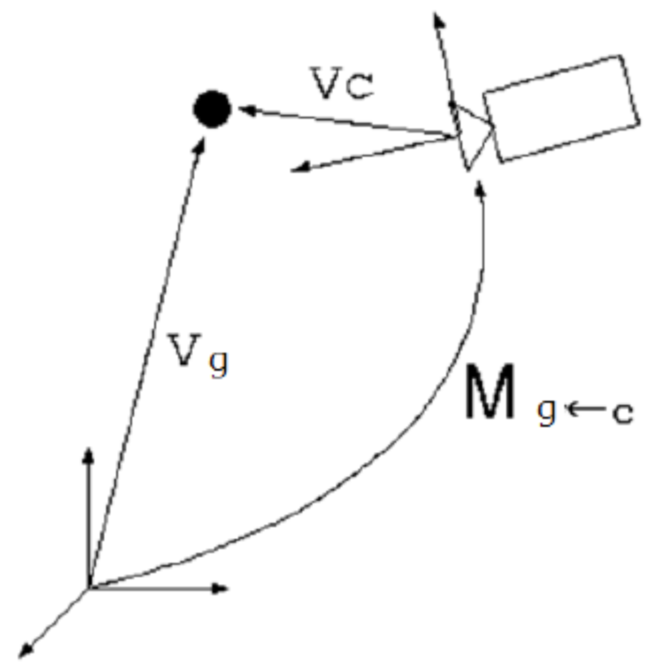
Viewing

- Now we have the world coordinates of all the vertices
- Now we want to convert the scene so that it appears in front of the camera



View Transformation

- We want to know the positions in the camera coordinate system
- We can compute the camera-to-world transformation matrix using the orientation and translation of the camera from the origin of the world coordinate system



View Transformation

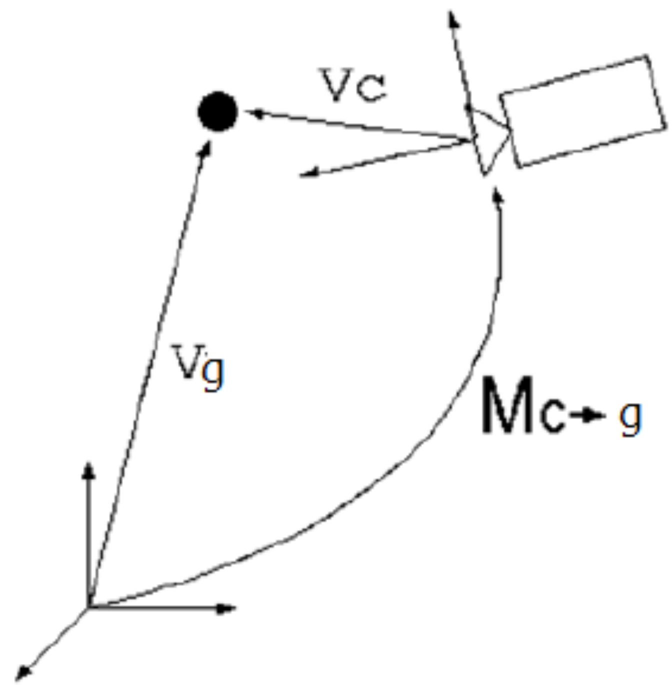
We want to know the positions in the camera coordinate system

$$vg = M_{g\leftarrow c}\ vc$$

Point in the world coordinate

Camera-to-world transformation

Point in the camera coordinate



$$vc = M^{-1}_{g\leftarrow c}\ vg$$

$$= M_{c\leftarrow g}\ vg$$

Summary

- Transformations: translation, rotation and scaling
- Using homogeneous transformation, 2D (3D) transformations can be represented by multiplication of a 3x3 (4x4) matrix
- Multiplication from left-to-right can be considered as the transformation of the coordinate system
- Reading: Shirley et al. Chapter6

  Foley et al. Chapter 5, Appendix 2 sections A1 to A5 for revision and further background (Chapter 5)

Lab session in the next lecture slot

Appleton Tower 5.05 from 11am-12pm on Friday 27th September

About using OpenGL and some description about the assignment

Modern Textbook

Shirley et al.