Curves and Surfaces 2

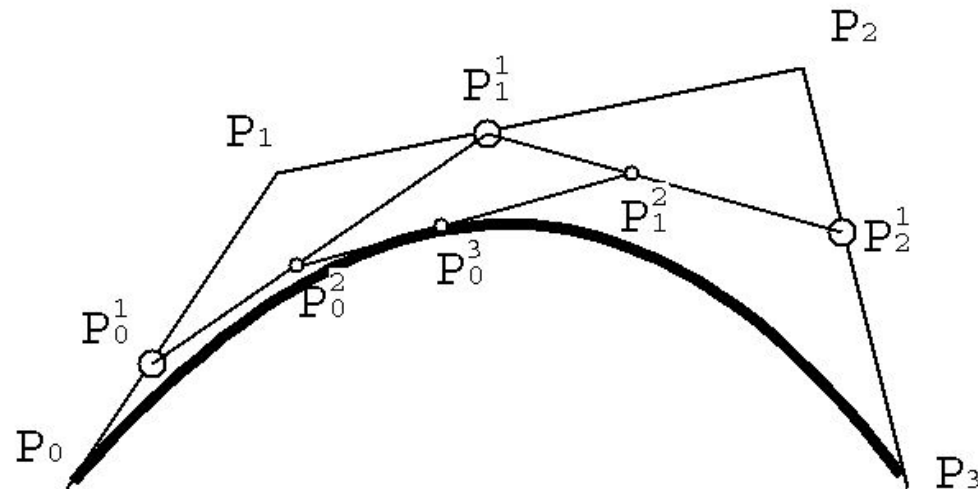## Computer Graphics

Lecture 17

Taku Komura

Today

- **More about Bezier and Bsplines**
    - **de Casteljau's algorithm**
    - BSpline : General form
    - de Boor's algorithm
   - Knot insertion
- NURBS
- Subdivision Surface

de Casteljau's Algorithm

- A method to evaluate (sample points in) or draw the Bezier curve
- The Bezier curve of any degree can be handled
- A precise way to evaluate the curves

de Casteljau's Algorithm

- Given the control points $P_1, ..., P_n$ and the parameter value $0 \le t \le 1$,
- Repeat the following procedure
  - Set $P^r_i(t) = (1 - t)\, P^{r-1}_i(t) + tP^{r-1}_{i+1}(t)$
  - $P^0_i(t) = P_i$
 - Then, $P^n_0(t)$ is the point with parameter value t on the Bezier curve
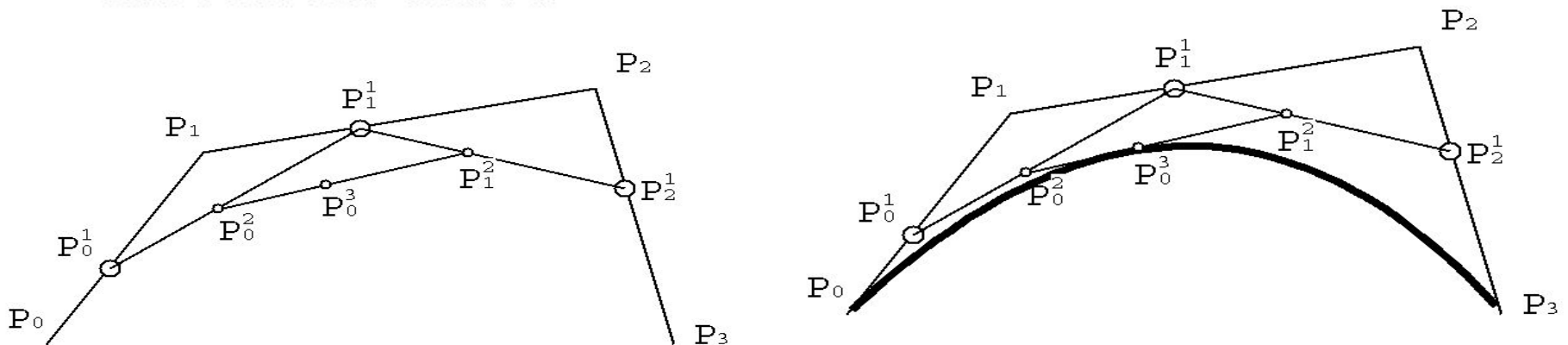
Why does this result in the polynomial?

- Let's think of a quadratic case that there are three points P_0, P_1, P_2.

- $P^1{}_0(t) = (1 - t)\,P_0 + tP_1$

- $P^1{}_1(t) = (1 - t)\,P_1 + tP_2$

- $P^2{}_0(t) = (1 - t)P^1{}_0 + tP^1{}_1$

- By inserting the first two equations into the third one, we obtain

- $P^2{}_0(t) = (1 - t)^2 P_0 + 2t(1 - t)P_1 + t^2 P_2$

- Doing this for 4 control points will give the cubic formula I presented last week

Why do we need this?

- The explicit representation (monomial form) that I presented last week can result in some instability
- Say the control points are randomly changed for 0.001.
- The curve computed by the de Casteljau's algorithm stays almost the same.
- The curve by the polynomial basis form can deviate from the original curve if the degree is high
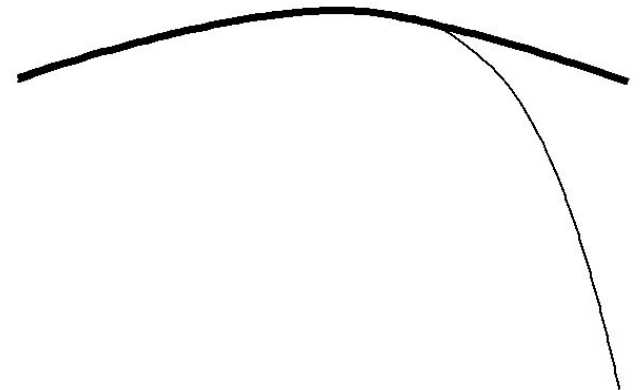
Connecting many Bezier Patches in the polynomial form

- The same story applies to surfaces
- The degree of surface can easily go high, as they are the multiplication of two curves
- Bicubic $\rightarrow$ 6
- The error of 16

control points will

be accumulated
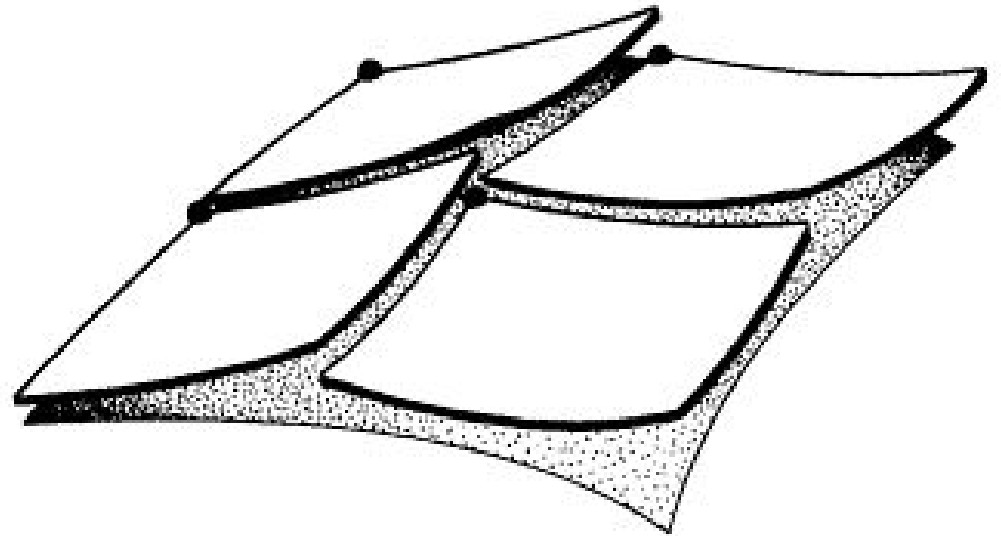
Today

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B-Spline : A General Form

A Bspline is a parametric curve composed of a linear combination of basis B-splines $B_{i,n}$

$$p(t) = \sum_{i=0}^{m} P_i B_{i,k}(t)$$

P_i $(i=0,...,m)$ the control points

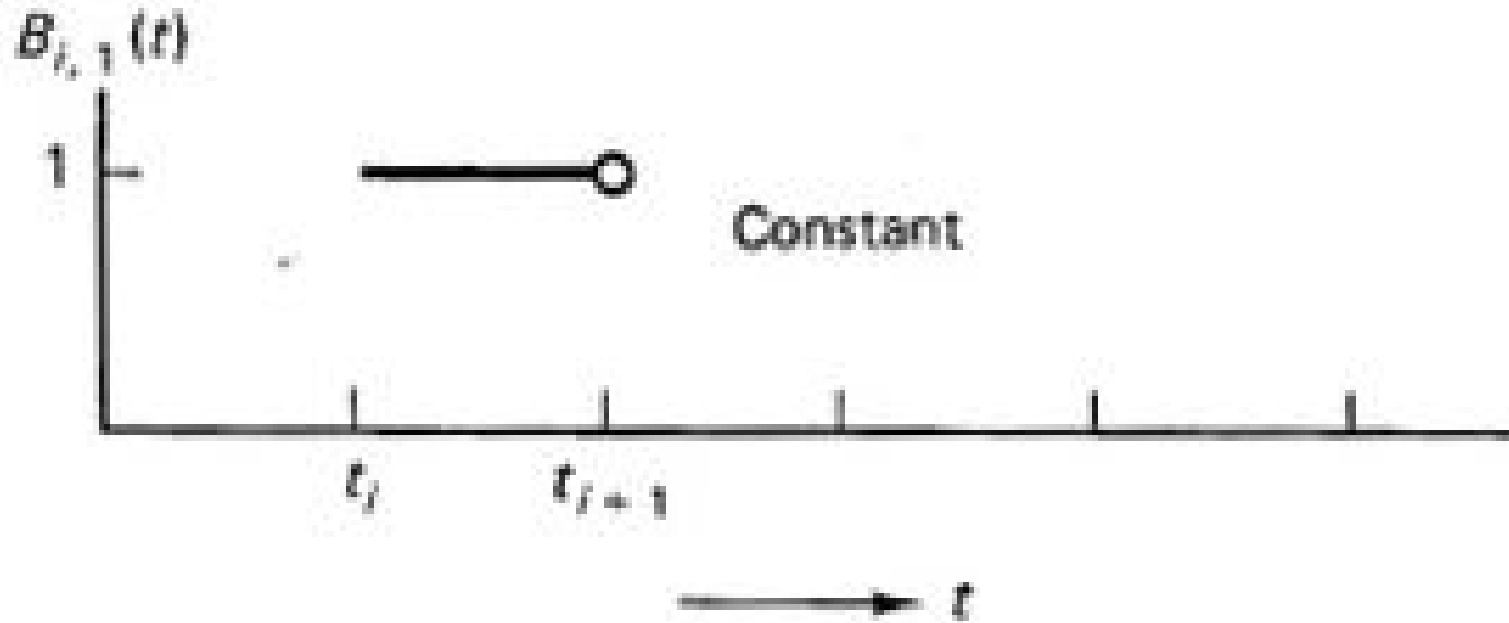**Knots:** $t_0 \le t_1 \le ... \le t_{k+m}$ - subdivide the domain of the B-spline curve into a set of knot spans $[t_i, t_{i+1})$
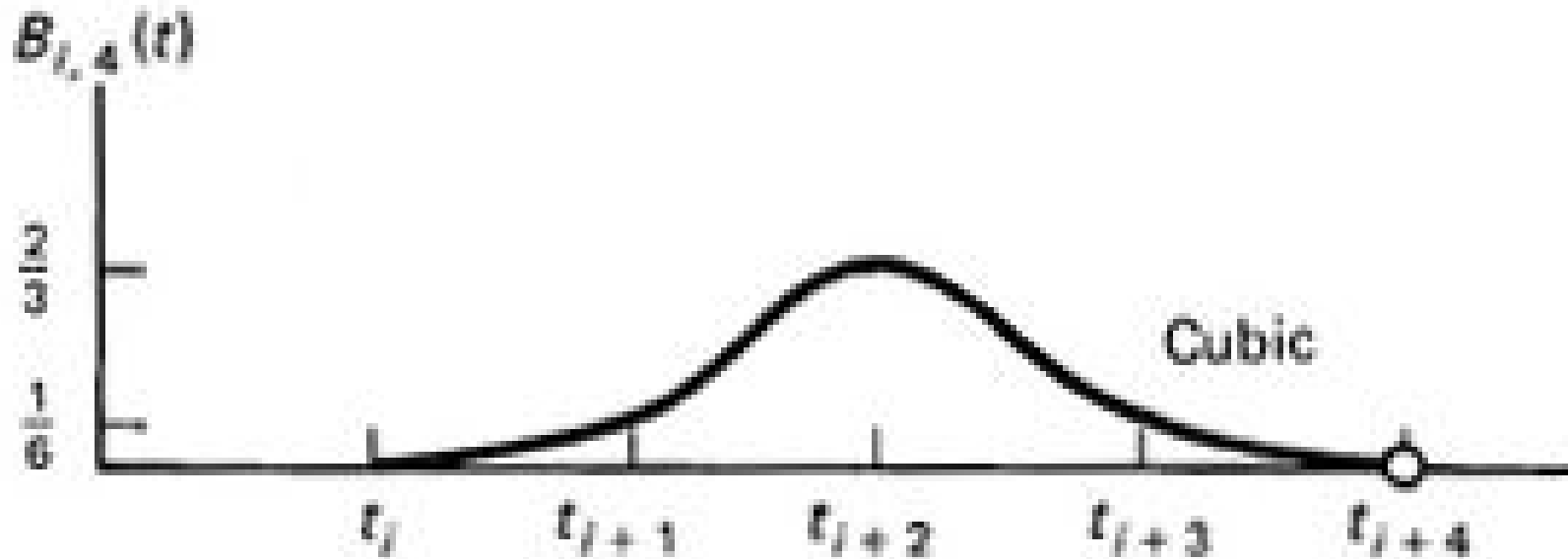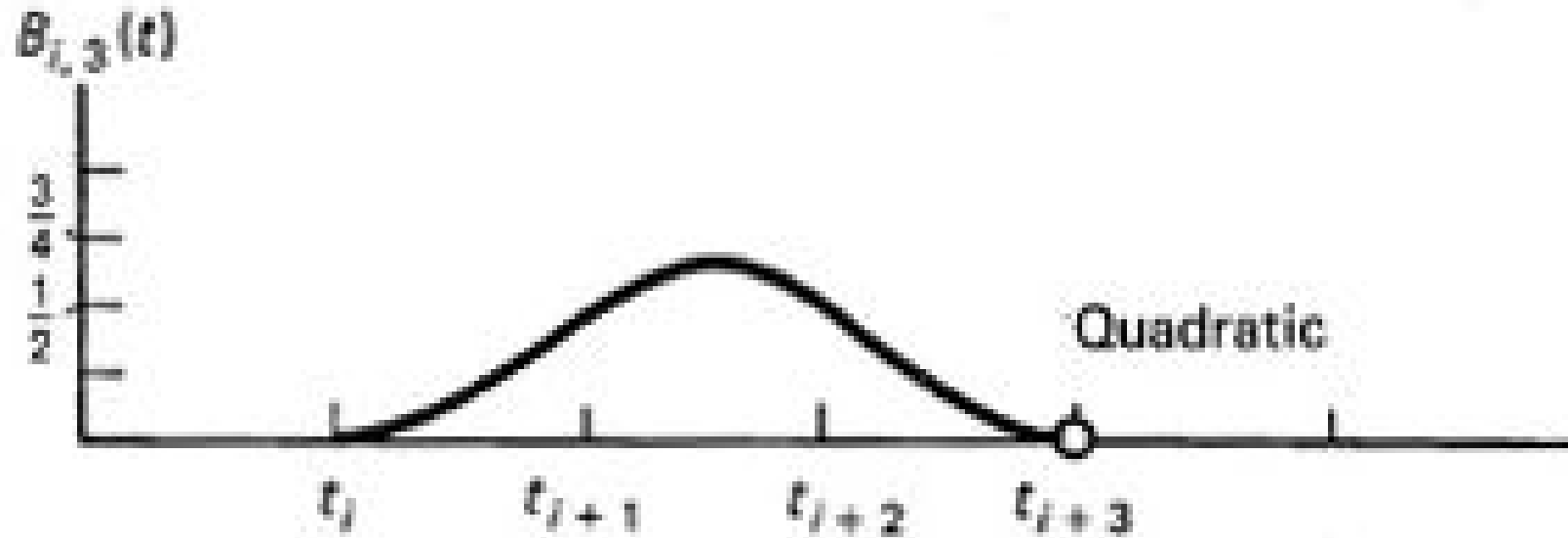
The B-splines can be defined by

$$B_{i,1}(t) = \begin{cases} 1, t_i \le t < t_{i+1} \\ 0, \text{otherwise} \end{cases}$$

$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k-1} - t_i} B_{i+1,k-1}(t)$$
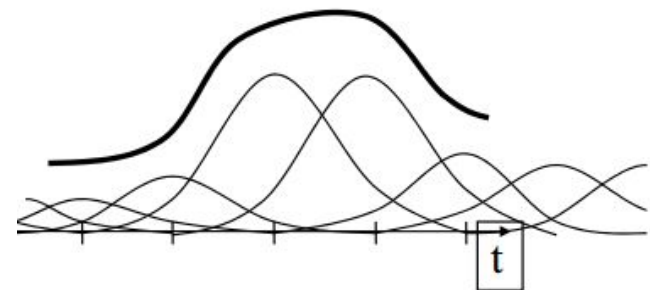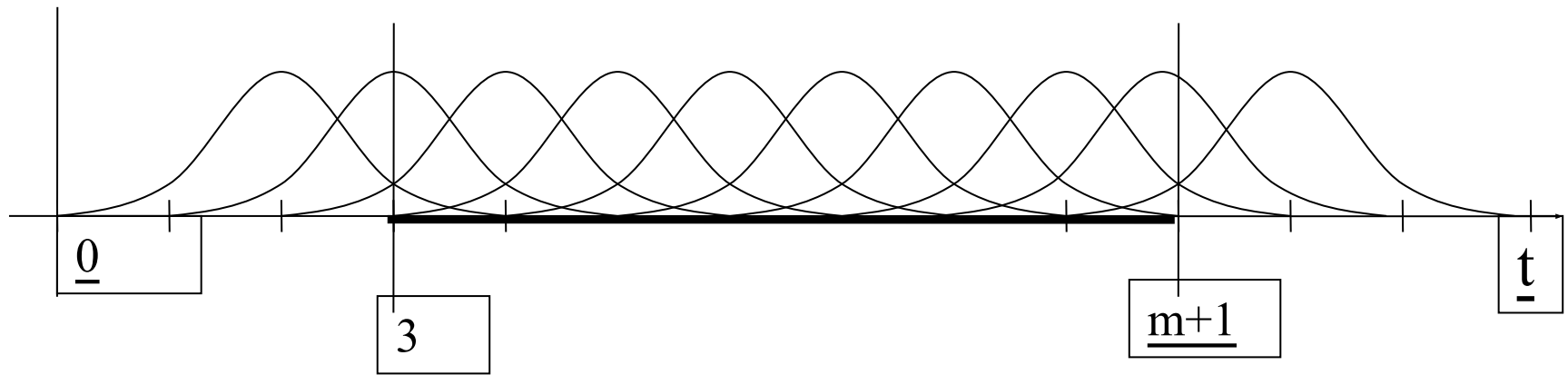
*

Bspline Basis
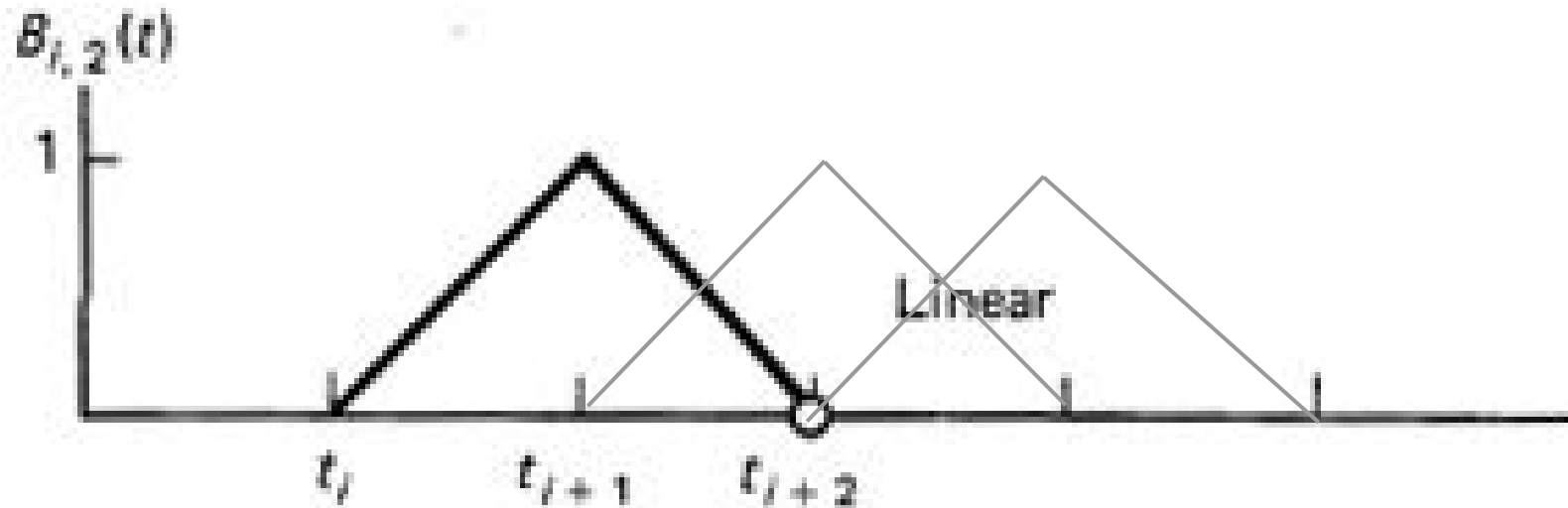
Bspline Basis (2)

Producing Curves by B-Splines

- The basis functions are multiplied to the control points and to define arbitrary curves
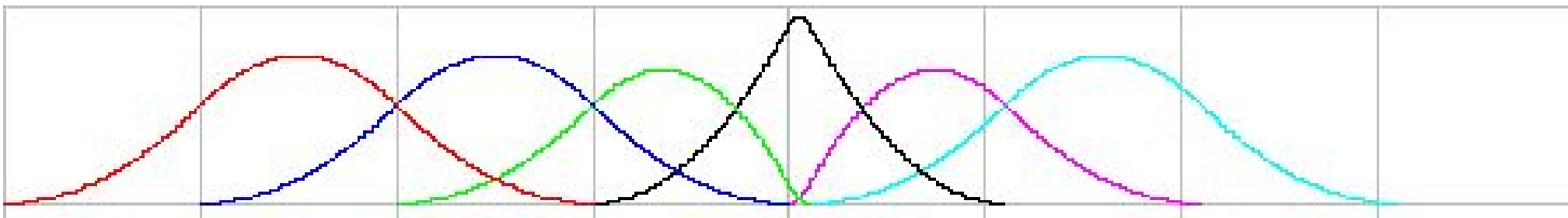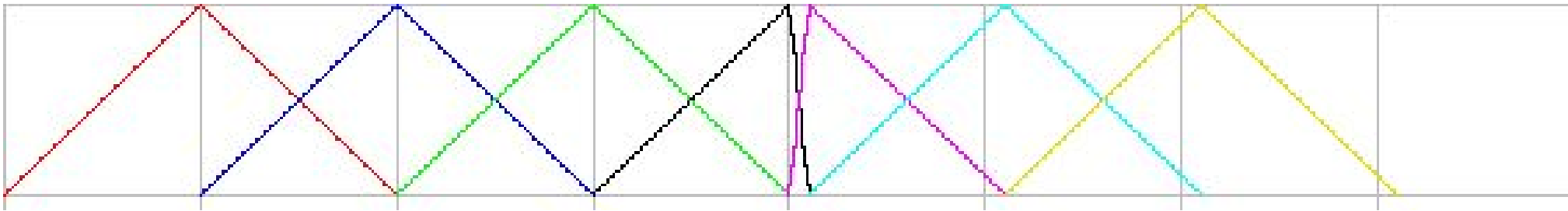
Knots

- The knots produce a vector that defines the domain of the curve
- The knots must be in the increasing order
- But not necessarily uniform
- If uniformly sampled and the degree is 3 uniform cubic bspline

| 0	1	2	3	4	5	6	7	8	9	10	11	12	13
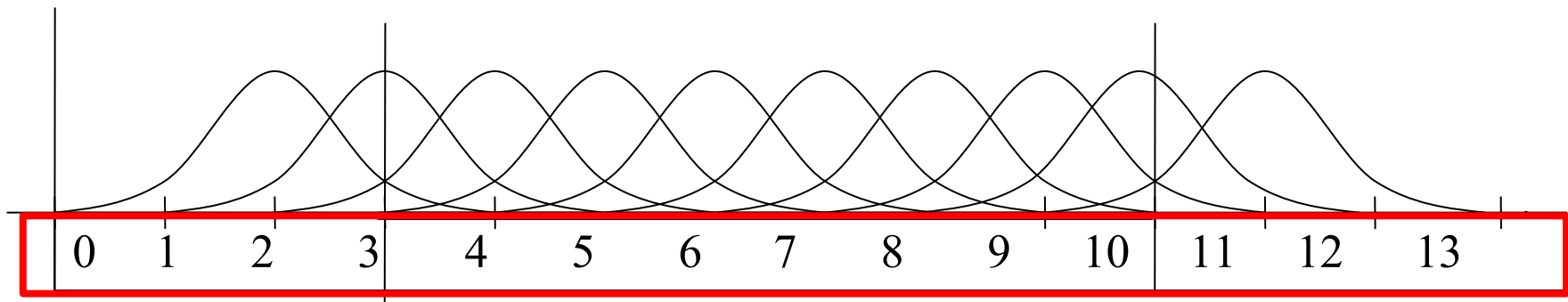
knots

Knots

Here is an example of non-uniform knots

Some Terms

- Order *k: the number of control points that affect the sampled value*
- Degree *k-1 (the basis functions are polynomials of degree k-1)*
- Control points **Pi (***i=0,...,m***)**
- Knots : *tj, (j=0,...., n)*
- An important rule :   n − m = k
- The domain of function $t_{k-1} \leqq t \leqq t_{m+1}$

  – Below, k = 4, m = 9,   domain, $t_3 \leqq t \leqq t_{10}$

0

3

m+1

t
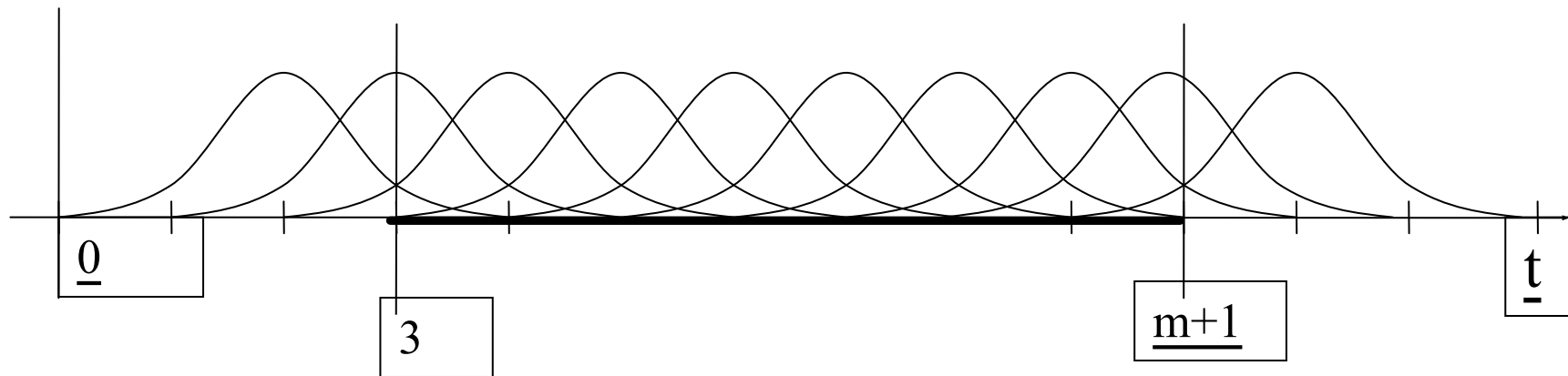
Clamped Bsplines

- The first and last knot values are repeated with multiplicity equal to the order (= degree + 1)
- The end points pass the control point
- For cubic Bsplines, the multiplicity of the first / last knots must be 4 (repeated four times)
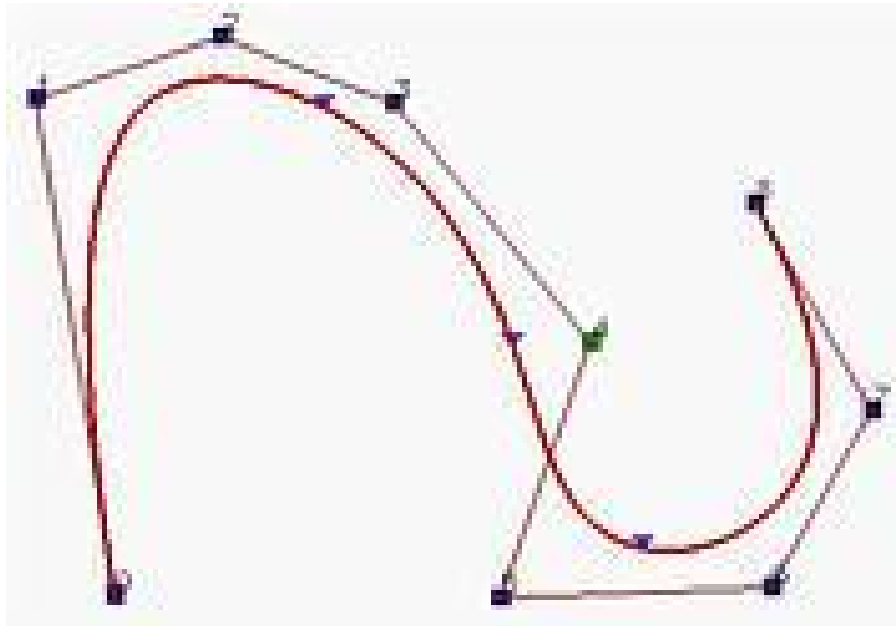
Controlling the shape of B-splines

- Moving the control points is the most obvious way to control B-spline curves
- Changing the position of control point $\mathbf{P}i$ only affects the local region
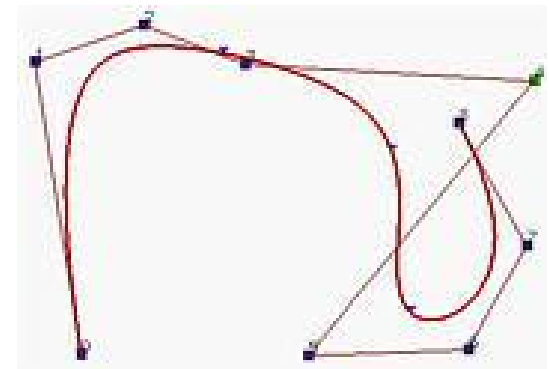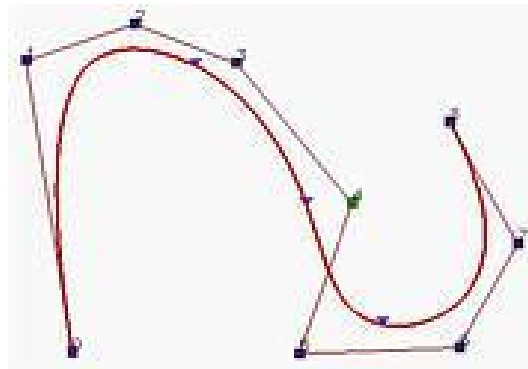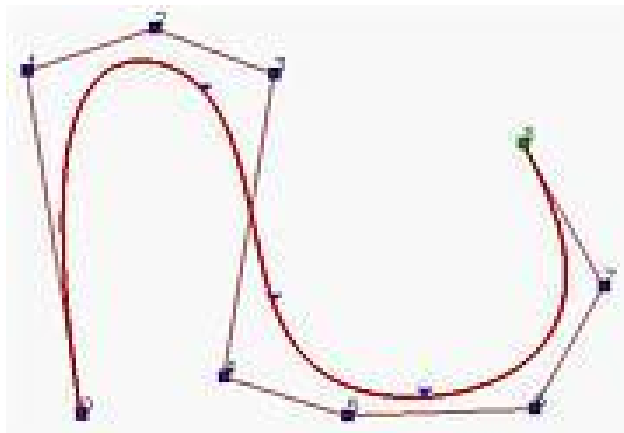
http://www.ibiblio.org/e-notes/Splines/basis.html

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- **More about Bezier and Bsplines**
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  - **de Boor's algorithm**
  - **Knot insertion**

- NURBS

- Subdivision Surface

De Boor's Algorithm

- A B-spline version of the de Casteljau's algo
- A precise method to evaluate the curve
- Starting from the control points $P_0, \dots, P_n$ and parameter value t, recursively solve the following problem
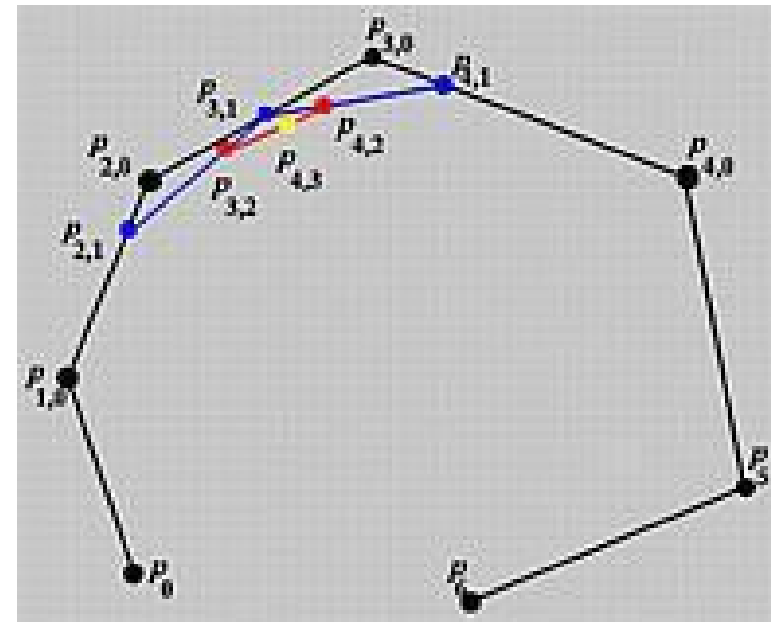
$$\mathbf{P}_i^r = (1 - a_{i,r})\mathbf{P}_{i-1}^{r-1} + a_{i,r}\mathbf{P}_i^{r-1}$$

$$a_{i,r} = \frac{t - t_i}{t_{i+k-1-r} - t_i}$$
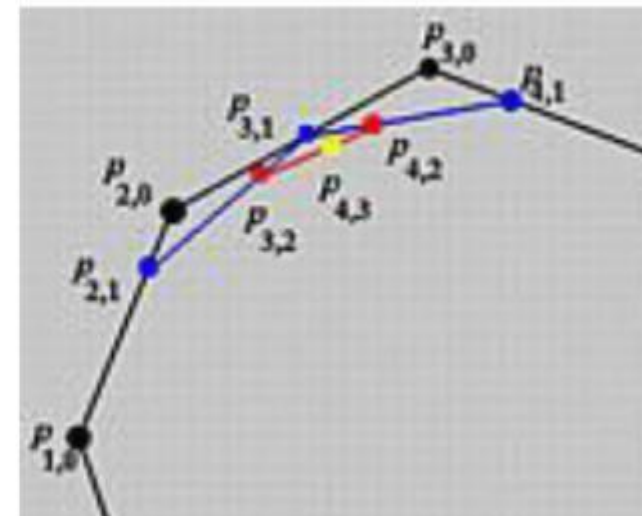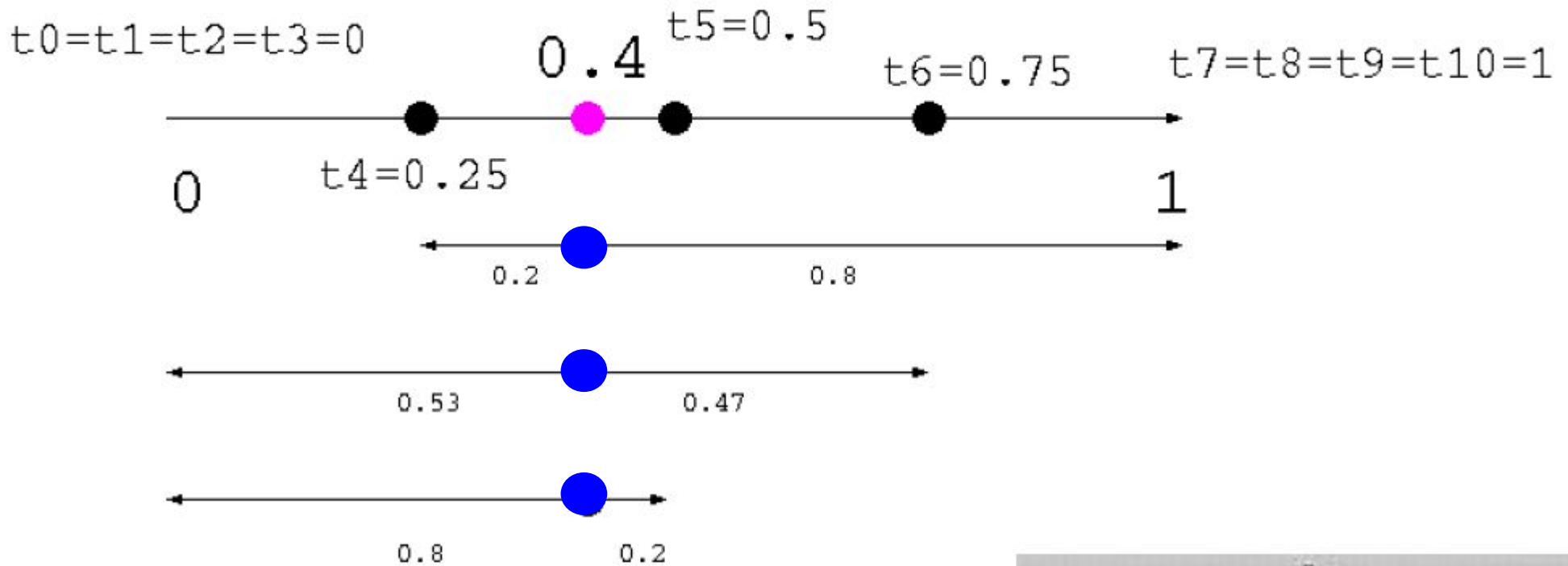
Example

- Assume we have a cubic B-spline whose knot vector is {0, 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1, 1}
- Let's compute a point at t = 0.4
- Then, $t_4 < t < t_5$, and the control points that affect the final position are P4, P3, P2, P1

Example

t0=t1=t2=t3=0
t5=0.5
0.4
t6=0.75
t7=t8=t9=t10=1

t4=0.25
0
1

0.2   0.8

0.53   0.47

0.8   0.2

$$a_{4,1} = (t - t_4)/(t_{4+3} - t_4) = 0.2$$

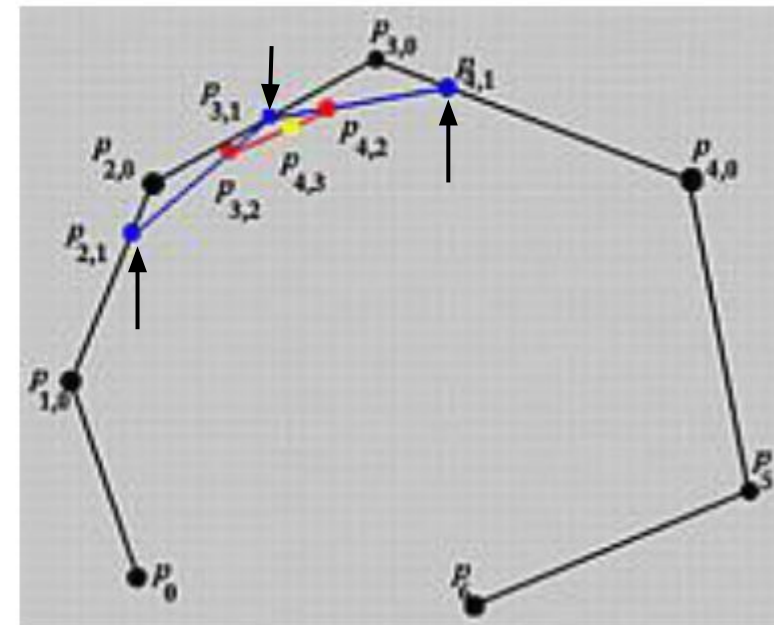$$a_{3,1} = (t - t_3)/(t_{3+3} - t_3) = 0.53$$

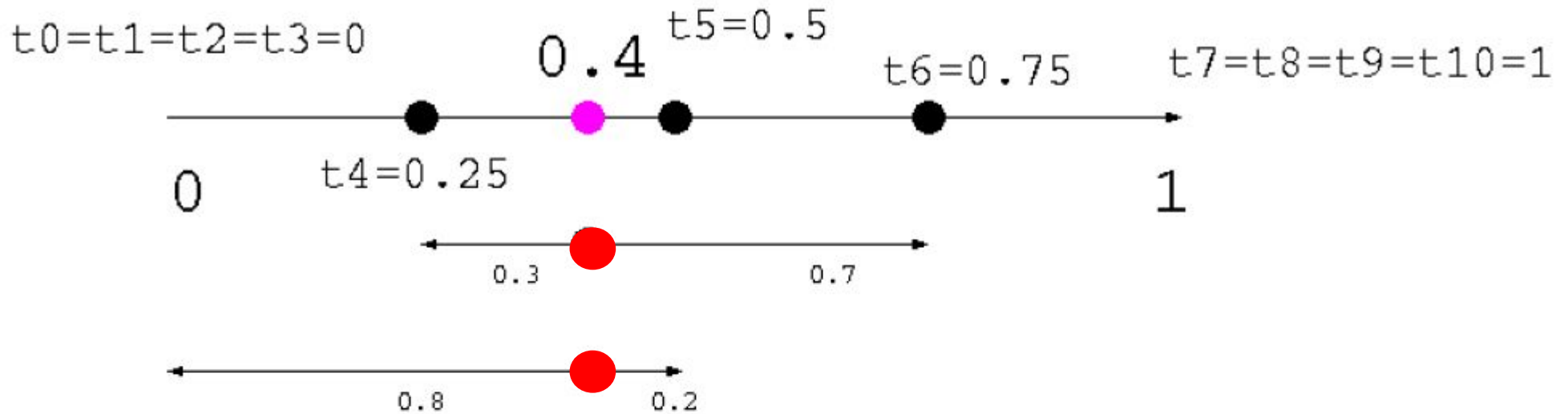$$a_{2,1} = (t - t_2)/(t_{2+3} - t_2) = 0.8$$

$$\mathbf{P}_{4,1} = (1 - a_{4,1})\mathbf{P}_{3,0} + a_{4,1}\mathbf{P}_{4,0}$$

$$\mathbf{P}_{3,1} = (1 - a_{3,1})\mathbf{P}_{2,0} + a_{3,1}\mathbf{P}_{3,0}$$

$$\mathbf{P}_{2,1} = (1 - a_{4,1})\mathbf{P}_{1,0} + a_{2,1}\mathbf{P}_{2,0}$$
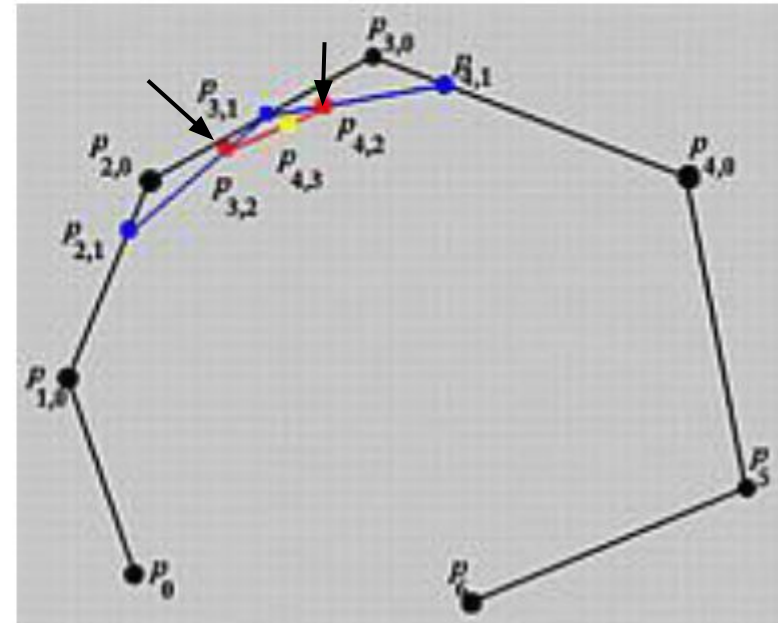
Example

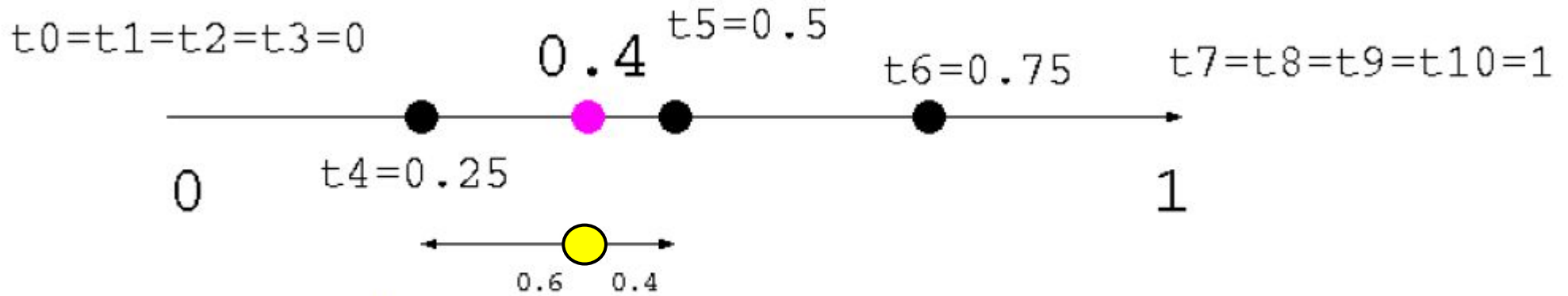$$a_{4,2} = (t - t_4) / (t_{4+3-1} - t_4) = 0.3$$

$$a_{3,2} = (t - t_3) / (t_{3+3-1} - t_3) = 0.8$$

$$\mathbf{P}_{4,2} = (1 - a_{4,2})\mathbf{P}_{3,1} + a_{4,2}\mathbf{P}_{4,1}$$

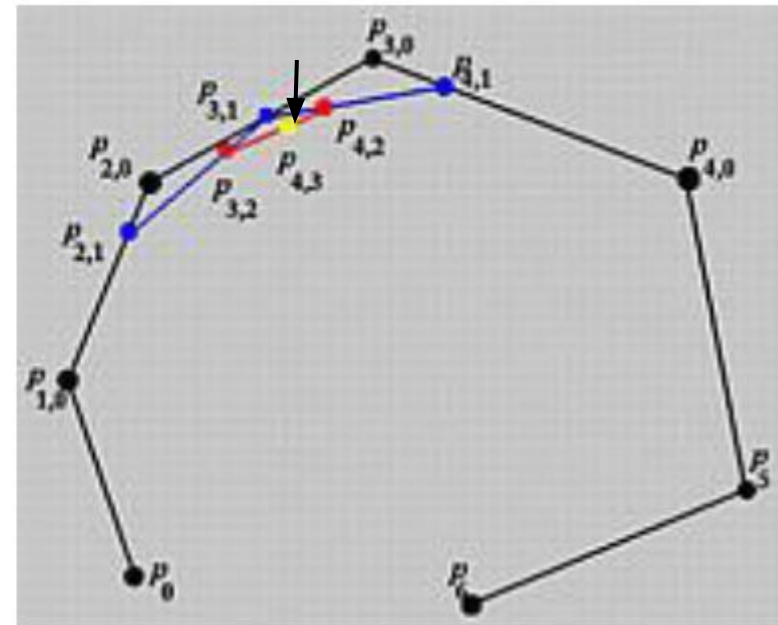$$\mathbf{P}_{3,2} = (1 - a_{3,2})\mathbf{P}_{2,1} + a_{3,2}\mathbf{P}_{3,1}$$

Example

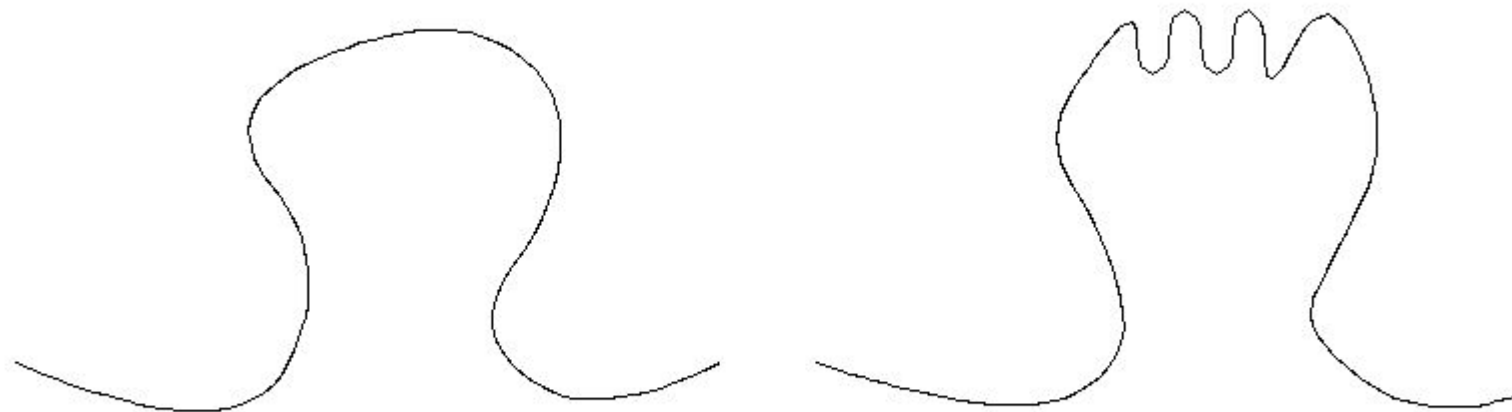$$a_{4,3} = (u - u_4)/(u_{4+3-2} - u_4) = 0.6$$

$$\mathbf{P}_{4,3} = (1 - a_{4,3})\mathbf{P}_{3,2} + a_{4,2}\mathbf{P}_{4,2}$$

$$= 0.4\mathbf{P}_{3,2} + 0.6\mathbf{P}_{4,2}$$
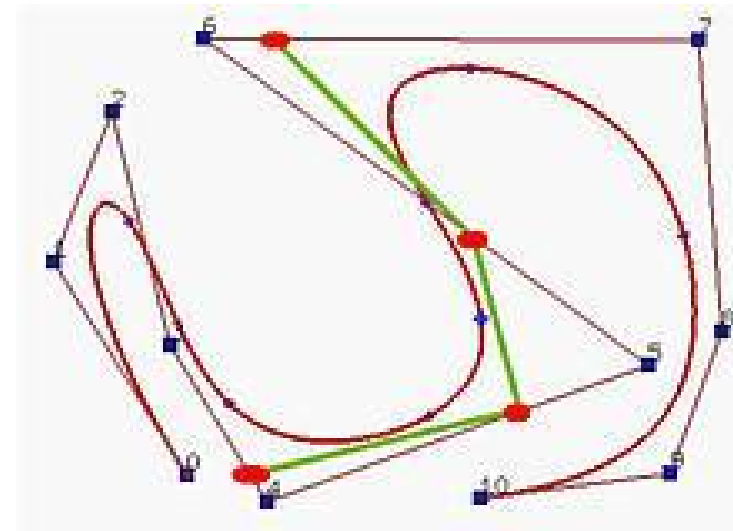
What if you want to edit some details?

- You might want to add some high resolution details at a particular area while keeping the rest the same
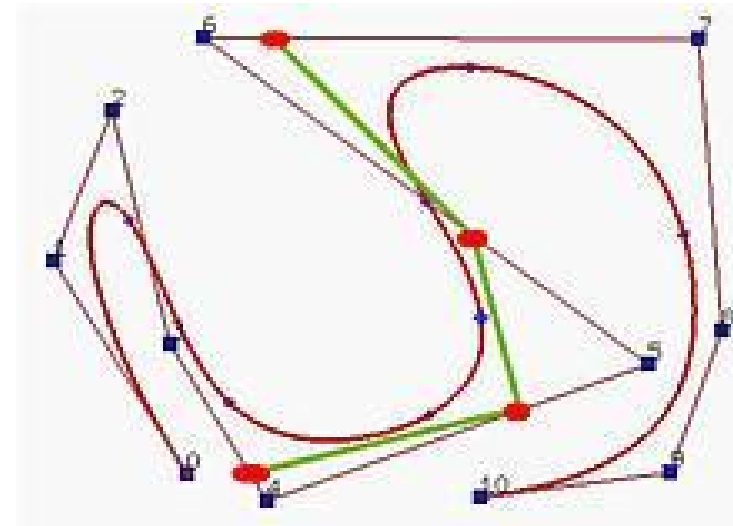
Knot insertion

- We can do this by knot Insertion
  - increase the resolution for some area
- New knots and control points can be added without changing the shape of the curve
- Both the knots and control points are going to increase

Knot insertion

- For a curve of degree f, we remove f-1 points and add f points
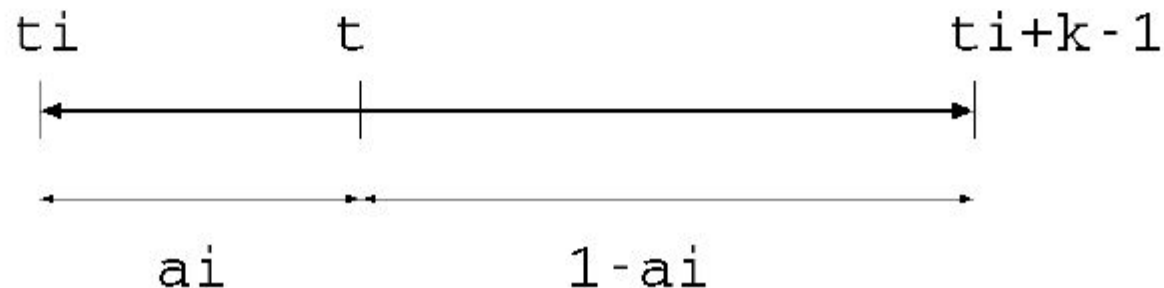- i.e., for a cubic B-spline, remove 2 points and add 3 points

Knot insertion

- If the new knot t is inserted into the span *[t_j, t_{j+1})*, the new control points can be computed by

$$\mathbf{Q_i} = (1 - a_i)\mathbf{P_{i-1}} + a_i\mathbf{P_i}$$

where **Qi** is the new control point and a_i is computed by

$$a_i = \frac{t - t_i}{t_{i+k-1} - t_i} \quad \text{for } j\text{-}k+2 \leq i \leq j$$

The polyline **P**j-k+1, **P**j-k+2, ..., **P**j-1, **P**j is replaced with **P**j-k+1, **Q**j-k+2, ..., **Q**j-1, **Q**j , **P**j.

```
     ti              t                    ti+k-1

      |--------------|---------------------|


      |--------------|---------------------|

           ai                 1-ai
```

Example

- A Bspline curve of degree 3 (k=4) having the following knots

- t=0.5 inserted

t_0 to t_3	t_4	t_5	t_6	t_7	t_8 to t_{11}
0	0.2	0.4	0.6	0.8	1

t_0 to t_3	t_4	t_5	t_6	t_7	t_8	t_9 to t_{12}
0	0.2	0.4	**0.5**	0.6	0.8	1

http://i33www.ira.uka.de/applets/mocca/html/no plugin/curves.html

Example

- A bspline curve of degree 3 (k=4) having the following knots

t_0 to t_3	t_4	t_5	t_6	t_7	t_8 to t_{11}
0	0.2	0.4	0.6	0.8	1

- t=0.5 inserted

t_0 to t_3	t_4	t_5	t_6	t_7	t_8	t_9 to t_{12}
0	0.2	0.4	**0.5**	0.6	0.8	1

$$a_5 = \frac{t - t_5}{t_8 - t_5} = \frac{0.5 - 0.4}{1 - 0.4} = \frac{1}{6}$$
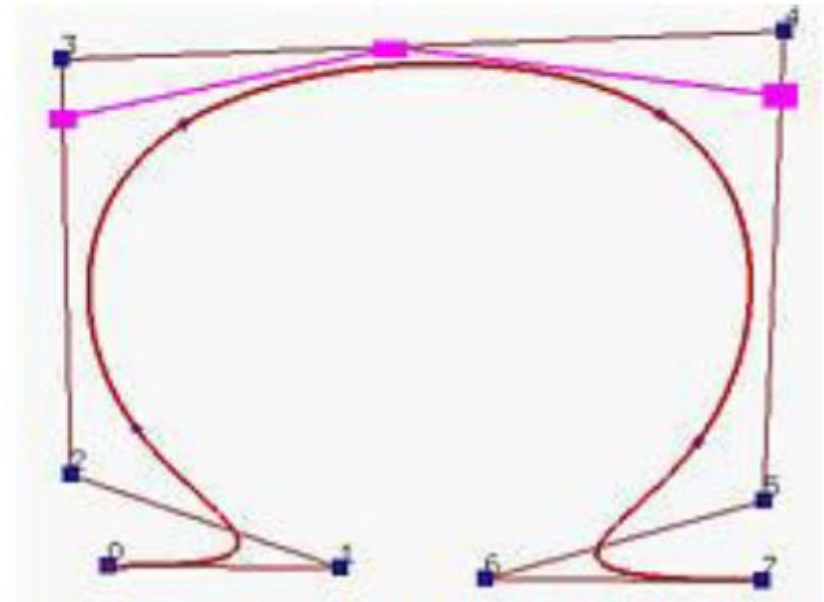
$$a_4 = \frac{t - t_4}{t_7 - t_4} = \frac{0.5 - 0.2}{0.8 - 0.2} = \frac{1}{2}$$

$$a_3 = \frac{t - t_3}{t_6 - t_3} = \frac{0.5 - 0}{0.6 - 0} = \frac{5}{6}$$

Example

- A bspline curve of degree 3 (k=4) having the following knots

t_0 to t_3	t_4	t_5	t_6	t_7	t_8 to t_{11}
0	0.2	0.4	0.6	0.8	1

- t=0.5 inserted

t_0 to t_3	t_4	t_5	t_6	t_7	t_8	t_9 to t_{12}
0	0.2	0.4	**0.5**	0.6	0.8	1

$$a_5 = \frac{t - t_5}{t_8 - t_5} = \frac{0.5 - 0.4}{1 - 0.4} = \frac{1}{6}$$

$$a_4 = \frac{t - t_4}{t_7 - t_4} = \frac{0.5 - 0.2}{0.8 - 0.2} = \frac{1}{2}$$

$$a_3 = \frac{t - t_3}{t_6 - t_3} = \frac{0.5 - 0}{0.6 - 0} = \frac{5}{6}$$

$$Q_5 = \left(1 - \frac{1}{6}P_4\right) + \frac{1}{6}P_5$$

$$Q_4 = \left(1 - \frac{1}{2}P_3\right) + \frac{1}{2}P_4$$

$$Q_3 = \left(1 - \frac{5}{6}P_2\right) + \frac{5}{6}P_3$$

Summary of B-splines

- Knot vector defines the domain
- Evaluation by de Boor's algorithm
- Controlling the shape by the control points
- Clamping the points by increasing the multiplicity of the knots at the end points
- Increase the resolution by knot insertion

Today

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  - ■ BSpline : General form
  - ■ de Boor's algorithm
  - ■ Knot insertion
- – **NURBS**
- – Subdivision Surface

NURBS (Non-uniform rational B-spline)

- Standard curves/surface representation in computer aided design

$$C(t) = \frac{\sum_{i=0}^{n} B_{i,k}(t) w_i \mathbf{P}_i}{\sum_{i=0}^{n} B_{i,k}(t) w_i}$$

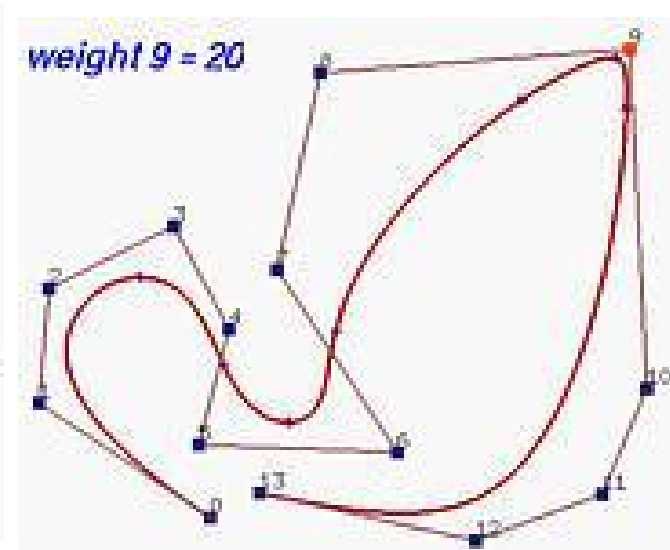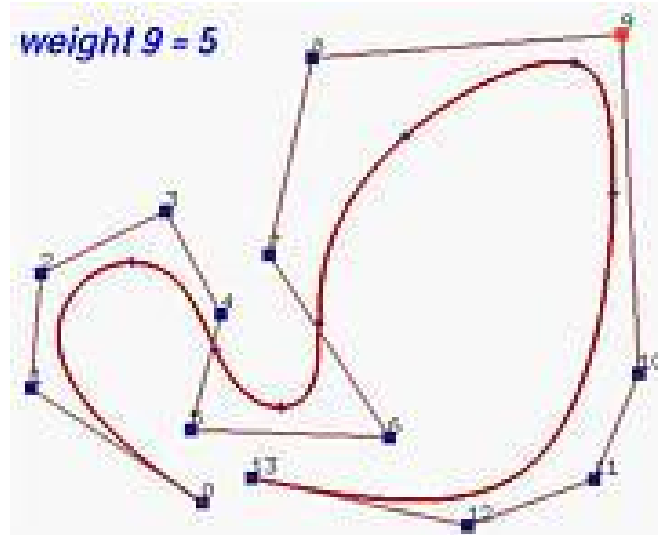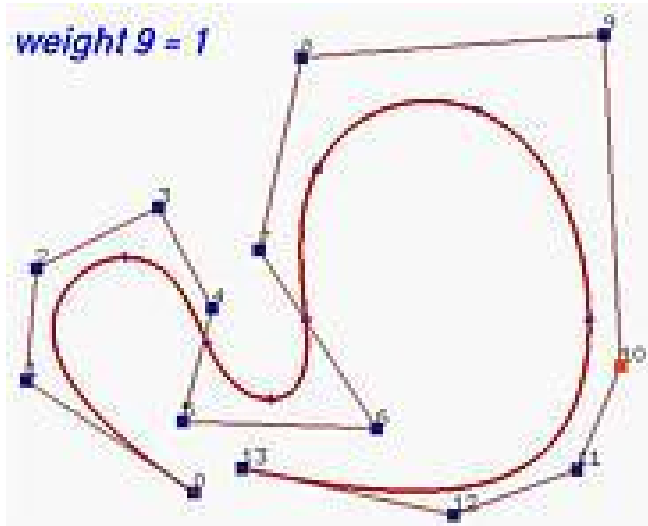**Pi** : control points

Bi,k: Bspline basis of order k

wi : weights

Benefits of using NURBS

- More degrees of freedom to control the curve (can control the weights)

- Invariant under perspective transformation
  - Can project the control points onto the screen and draw the NURBS on the screen
    - Don't need to apply the perspective transformation to all the sample-points on the curve

- Can model conic sections such as circles, ellipses and hyperbolas

Example of changing weights

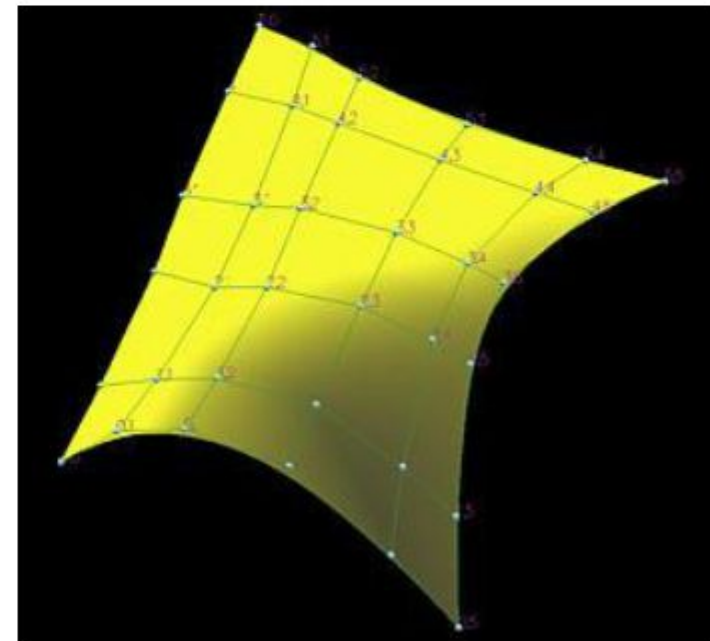- Increasing the weight will bring the curve closer to the corresponding control point

Bspline Surfaces

- Given the following information:
- a set of $m+1$ rows and $n+1$ control points $\mathbf{p}i,j$, where $0 <= i <= m$ and $0 <= j <= n$;
- Corresponding knot vectors in the u *and* v direction,

$$p(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_{i,p}(u)B_{j,q}(v)\mathbf{P}_{i,j} : \text{non}-\text{rational B-spline}$$

$$p(u,v) = \frac{\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_{i,p}(u)B_{j,q}(v)\mathbf{P}_{i,j}}{\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_{i,p}(u)B_{j,q}(v)} : \text{NURBS}$$

Clamping, Closing B-spline Surfaces

- The B-spline surfaces can be clamped by repeating the same knot values in one direction of the parameters (u or v)
- We can also close the curve / surface by recycling the control points
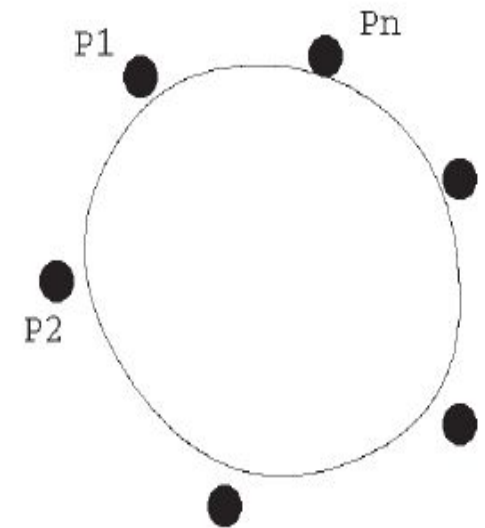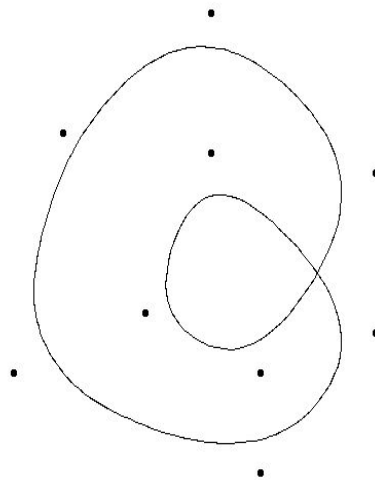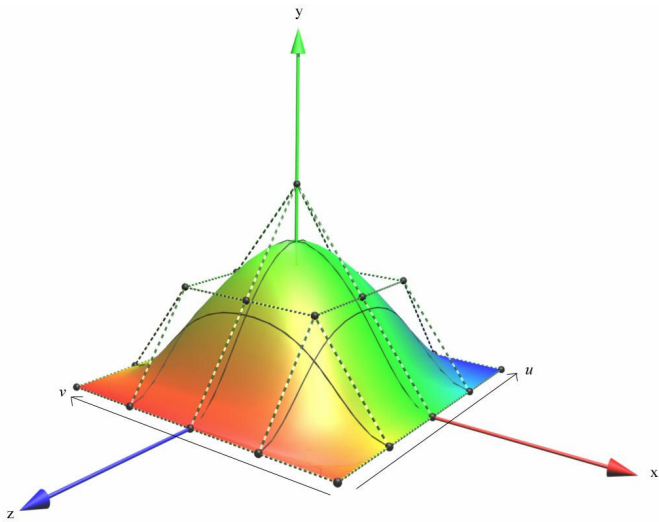
Closed B-spline Surfaces

- If a B-spline surface is closed in one direction, then the surface becomes a tube.
- Closed in two direction : torus
  - Problems handling objects of arbitrary topology, such as a ball, double torus
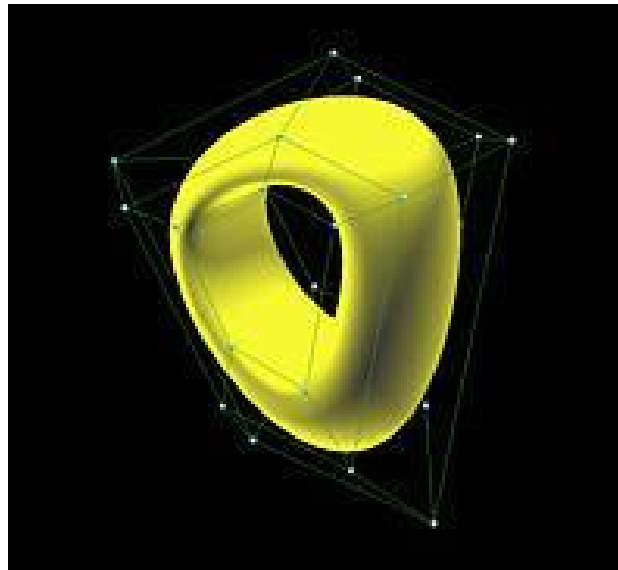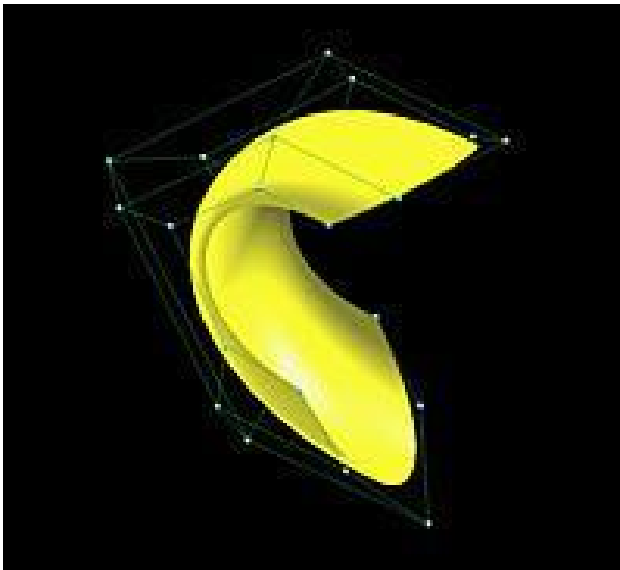
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  - de Casteljau's algorithm
  - BSpline : General form
  - de Boor's algorithm
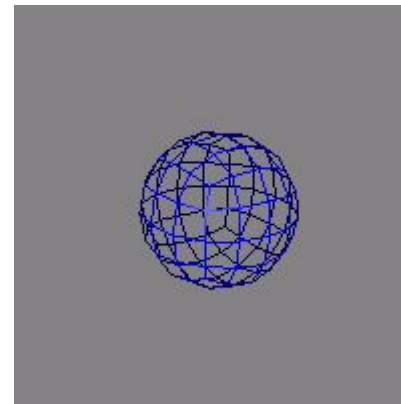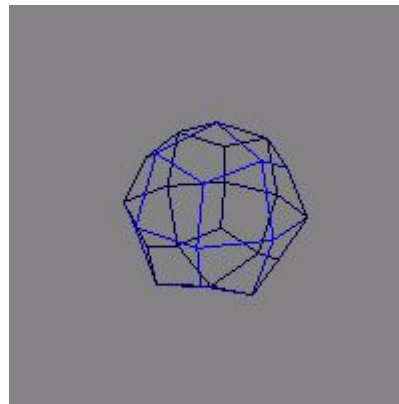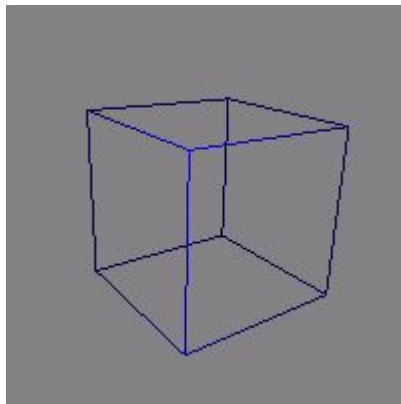 - Knot insertion
- NURBS
- **Subdivision Surface**
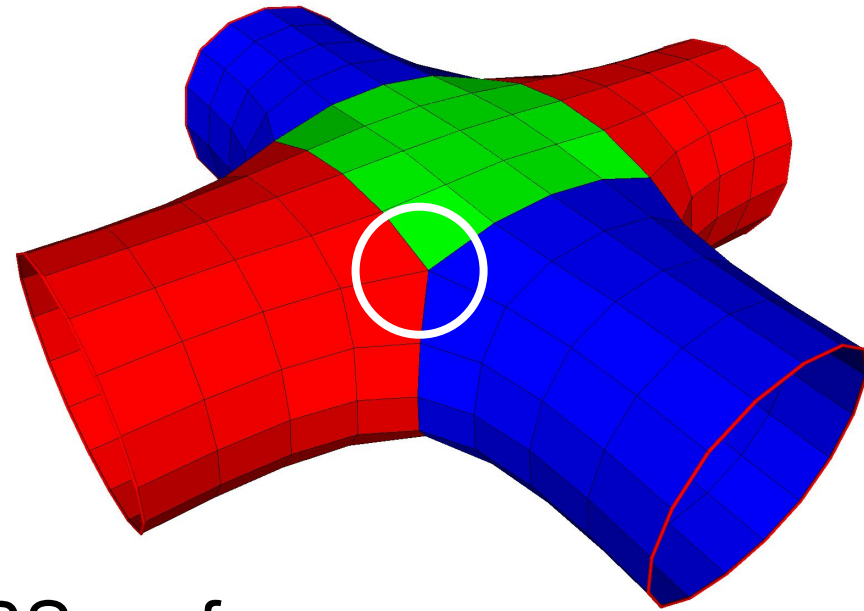
Subdivision Surface

- A method to model smooth surfaces

3D subdivision surface

- Giving a rough shape first and subdivide it recursively
- Stop when the shape is smooth enough
- Used for modeling smooth surfaces

Motivation

- ## Shape modelling
  - Topological restrictions of NURBS surfaces
    - Plane, Cylinder, and Torus
    - It is difficult to maintain smoothness at seams of patchwork.
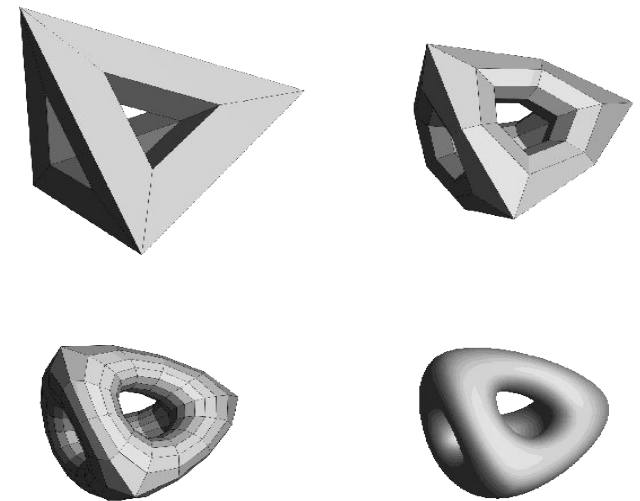   - Example: hiding seams in Woody (*Toy Story*) [DeRose98]
  - NURBS also require the control nets consist of a regular rectangular grid of control points
- ## LOD in a scene
  - A coarse shape when far away, a smooth dense surface when closer to the camera
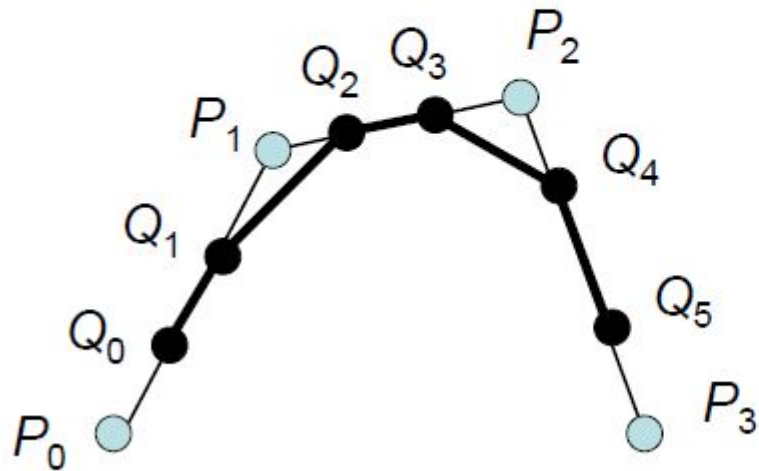
Subdivision surface
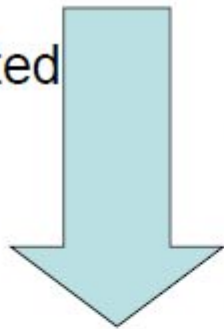
- Can handle arbitrary topology

Different Schemes

- Doo-Sabin '78
- Catmull-Clark '78
- Etc (Loop, Butterfly, and many others)
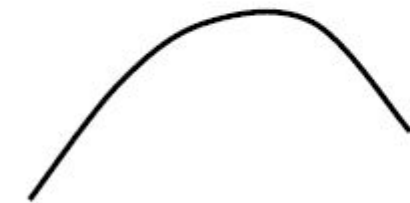
A Primer: Chaiken's Algorithm

Apply Iterated Function System

$$Q_{2i} = \frac{3}{4}P_i + \frac{1}{4}P_{i+1}$$

$$Q_{2i+1} = \frac{1}{4}P_i + \frac{3}{4}P_{i+1}$$

Limit Curve Surface

$$Q_0 = \frac{3}{4}P_0 + \frac{1}{4}P_1$$

$$Q_1 = \frac{1}{4}P_0 + \frac{3}{4}P_1$$

$$Q_2 = \frac{3}{4}P_1 + \frac{1}{4}P_2$$
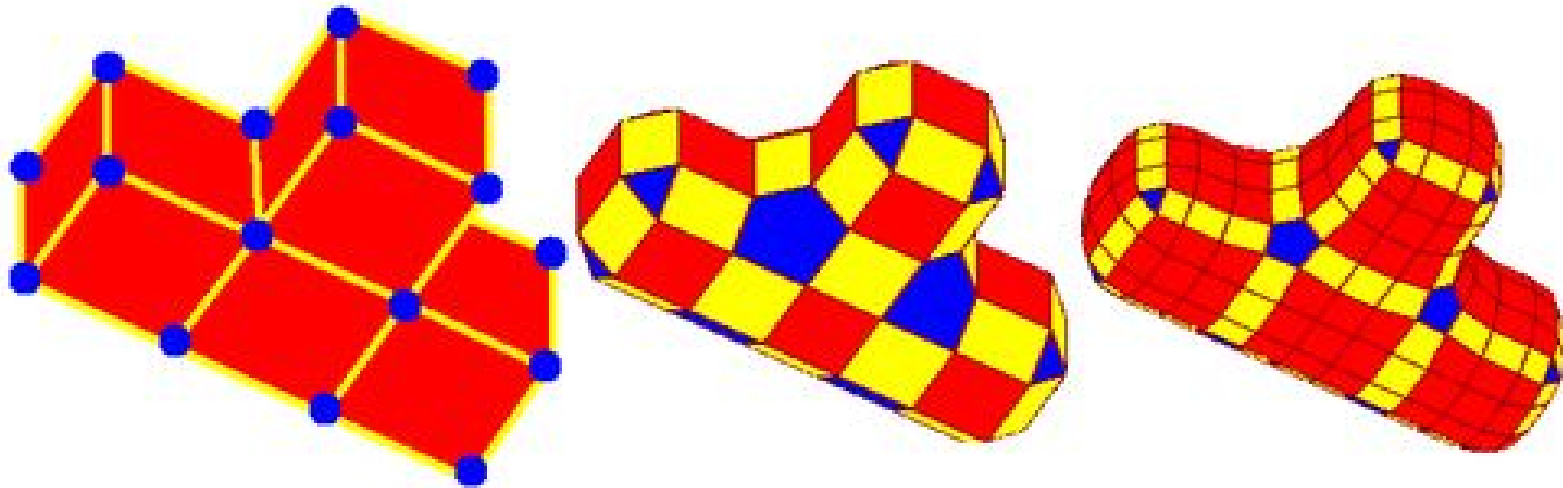
$$Q_3 = \frac{1}{4}P_1 + \frac{3}{4}P_2$$

$$Q_4 = \frac{3}{4}P_2 + \frac{1}{4}P_3$$

$$Q_5 = \frac{1}{4}P_2 + \frac{3}{4}P_3$$

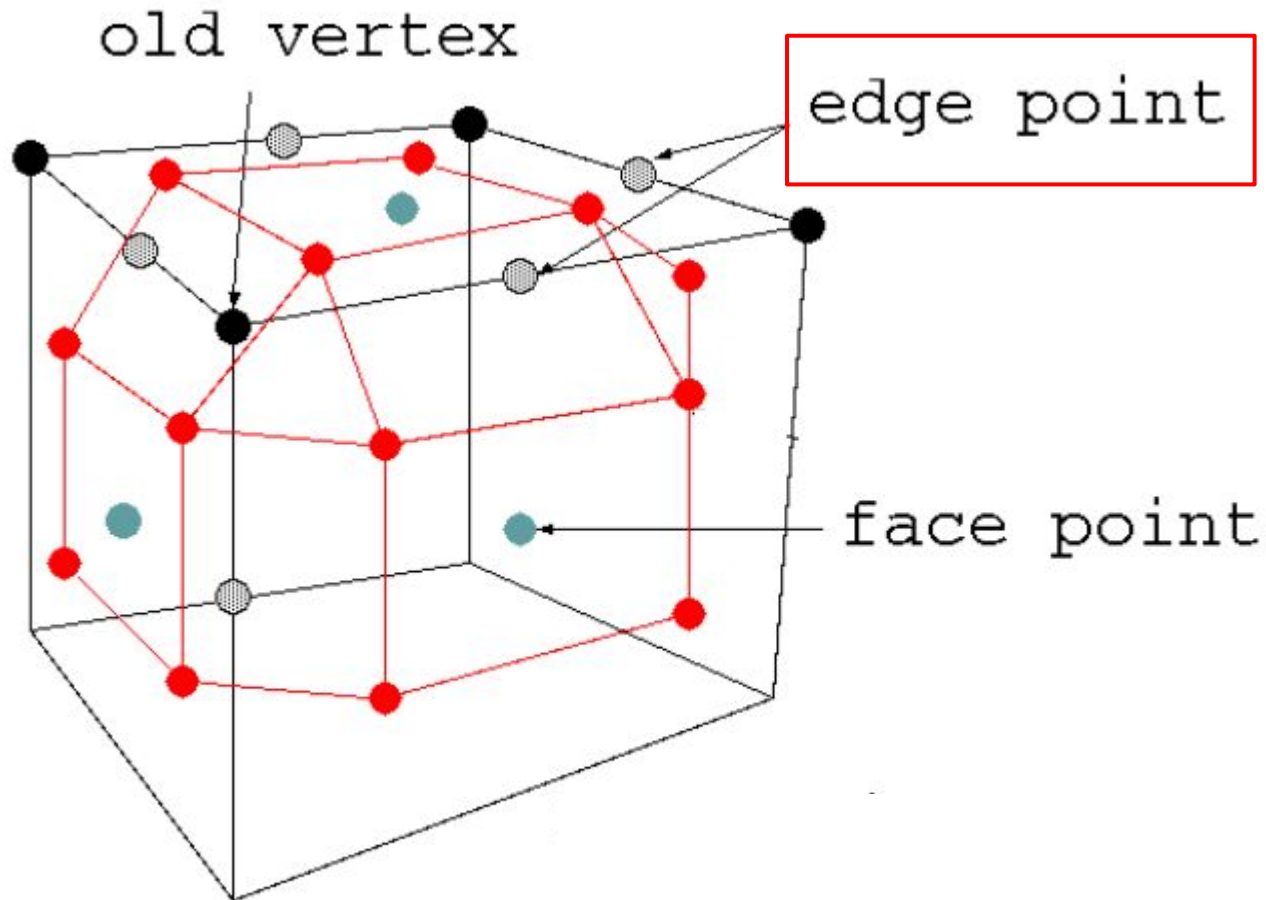http://www.multires.caltech.edu/teaching/demos/java/chaikin.htm

Doo-Sabin Subdivision

- Proposed by Doo and Sabin in 1978
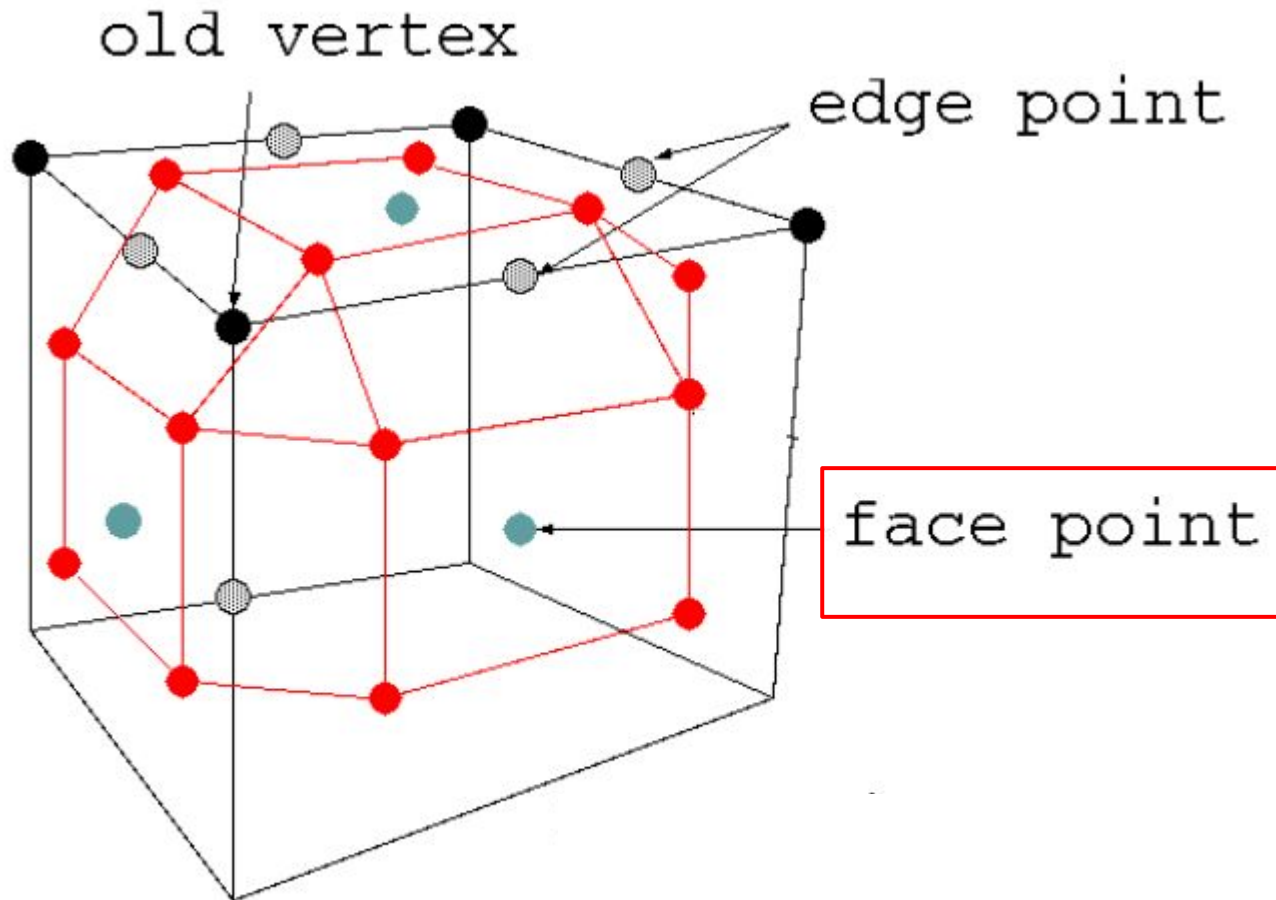- An extension of Chaiken's algorithm to 3D mesh surfaces

Doo-Sabin Subdivision (2)

- **An *edge point* is formed from the midpoint of each edge**
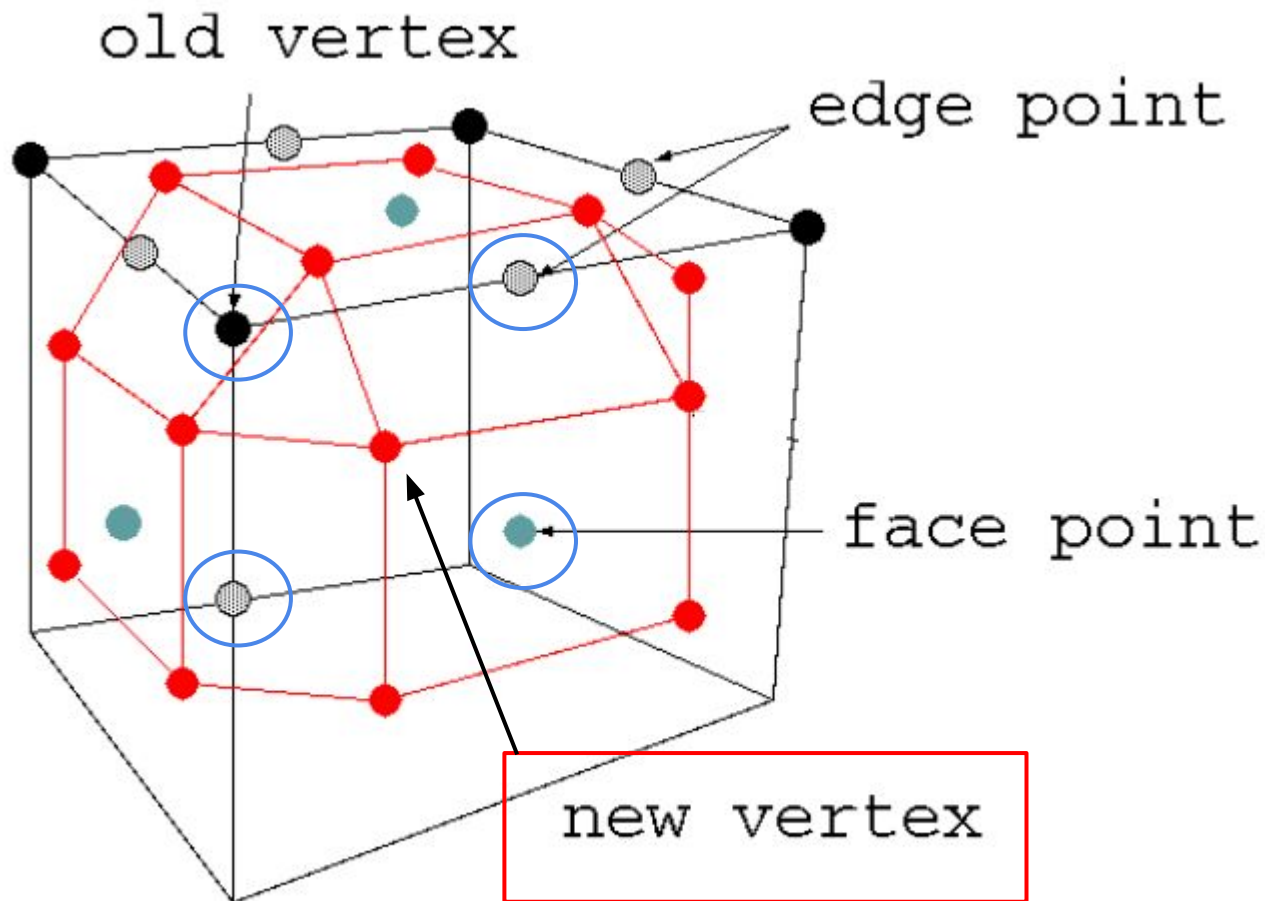- A *face point* is formed as the centroid of each polygon of the mesh.

Doo-Sabin Subdivision (2)

- An *edge point* is formed from the midpoint of each edge
- **A *face point* is formed as the centroid of each polygon of the mesh.**
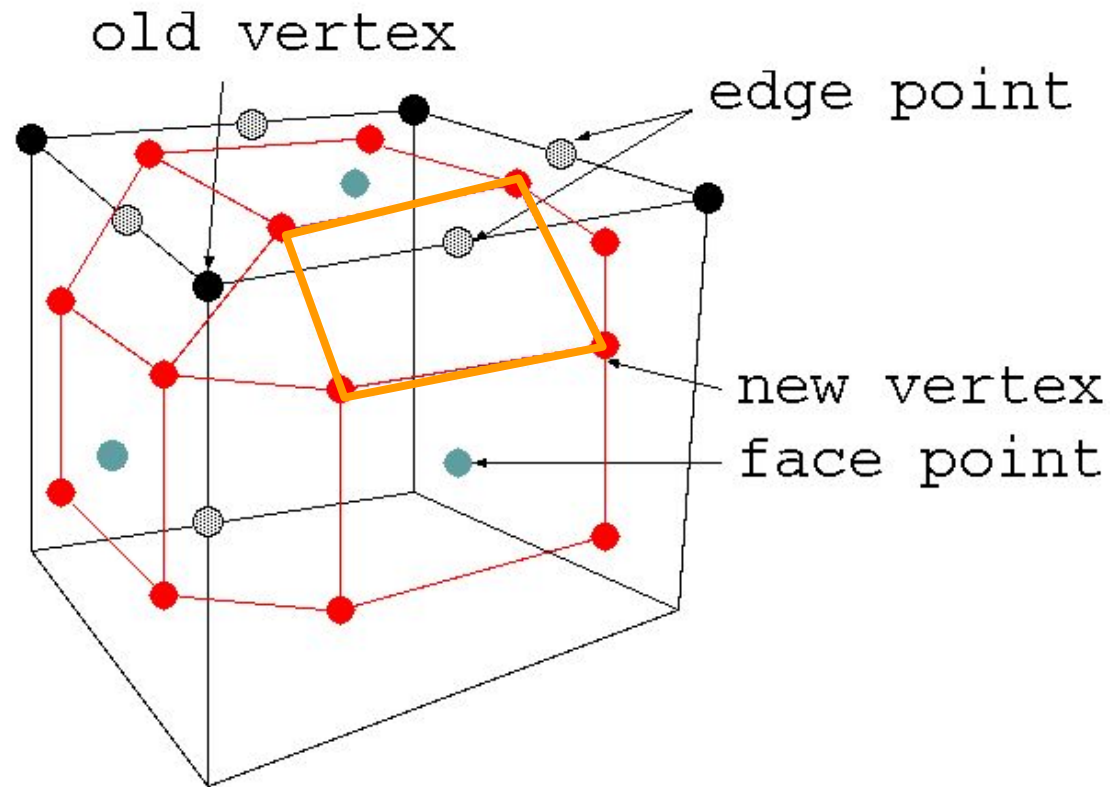
old vertex

edge point

face point

Doo-Sabin Subdivision (2)

- Finally, each vertex in new mesh is formed as average of
  - a vertex in the old mesh,
  - a face point for a polygon that touches that old vertex, and
  - the edge points for the two edges that belong to that polygon and touch that old vertex.
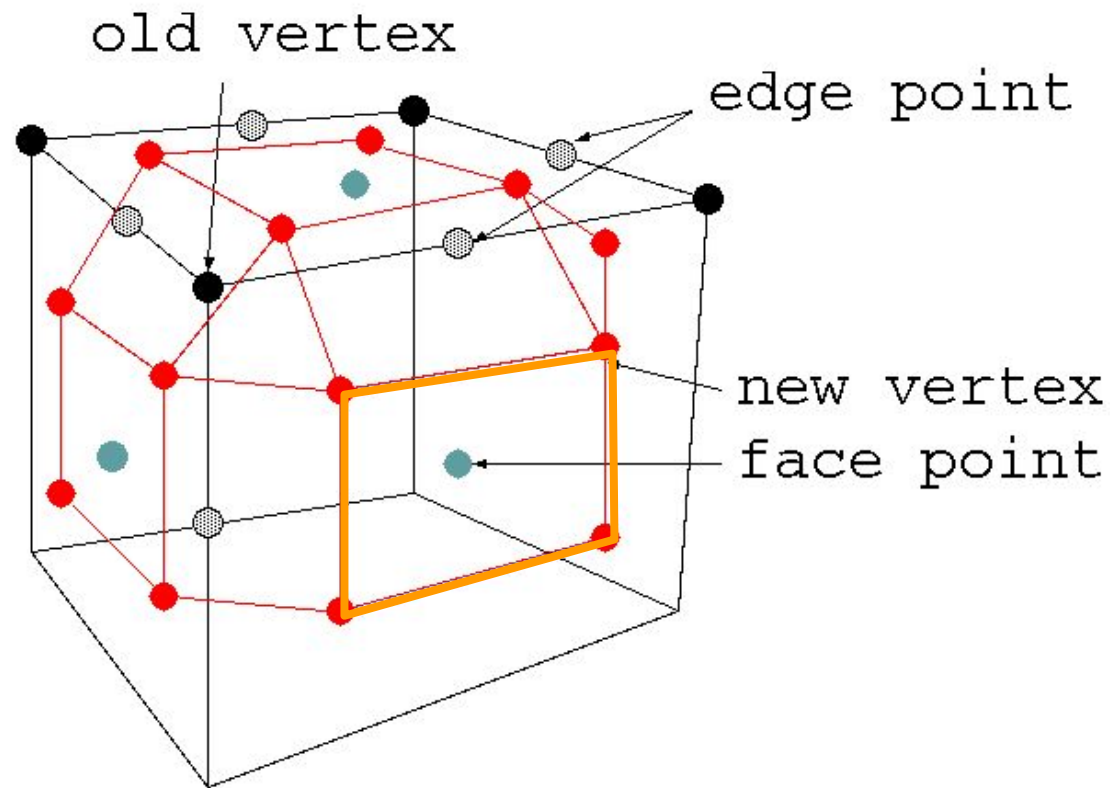
Doo-Sabin Subdivision (3)

The new mesh will be composed of
- **quadrilaterals for each edge in the old mesh,**
- a smaller n-sided polygon for each n-sided polygon in the old mesh, and
- an n-sided polygon for each n-valence vertex (Valence being the number of edges that touch the vertex).

Doo-Sabin Subdivision (3)

old vertex

edge point

new vertex
face point

The new mesh will be composed of
- quadrilaterals for each edge in the old mesh,
- **a smaller n-sided polygon for each n-sided polygon in the old mesh, and**
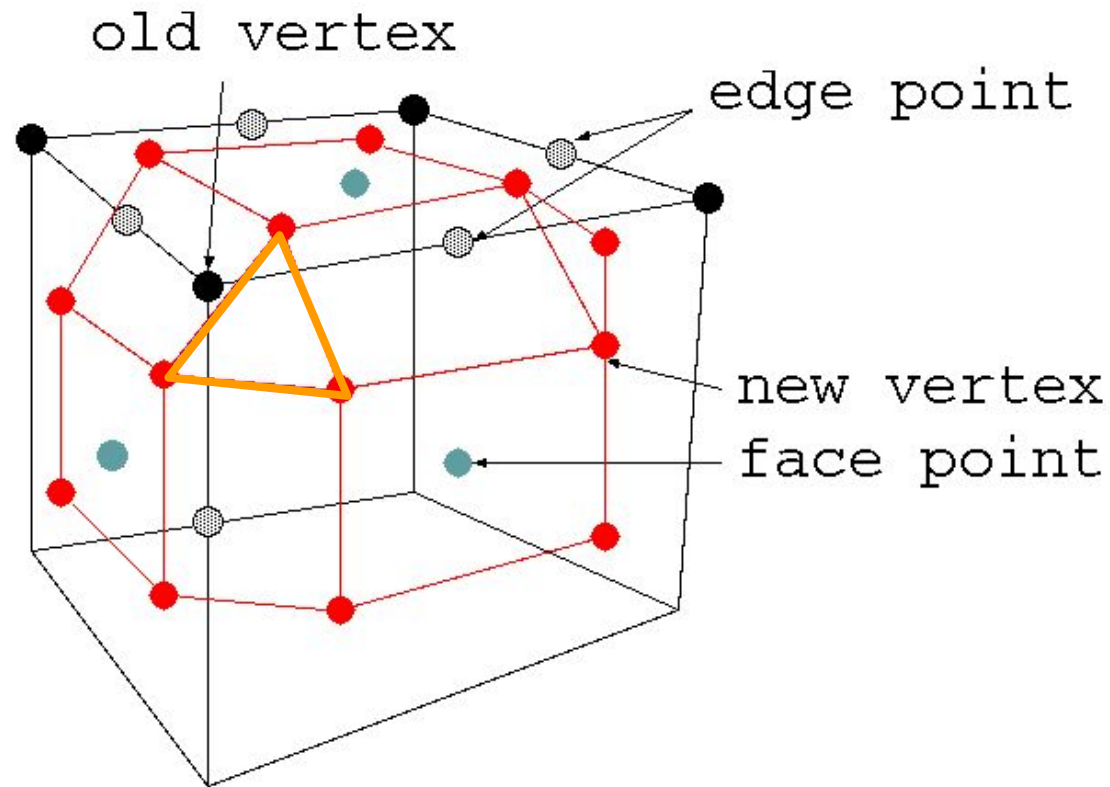- an n-sided polygon for each n-valence vertex (Valence being the number of edges that touch the vertex).
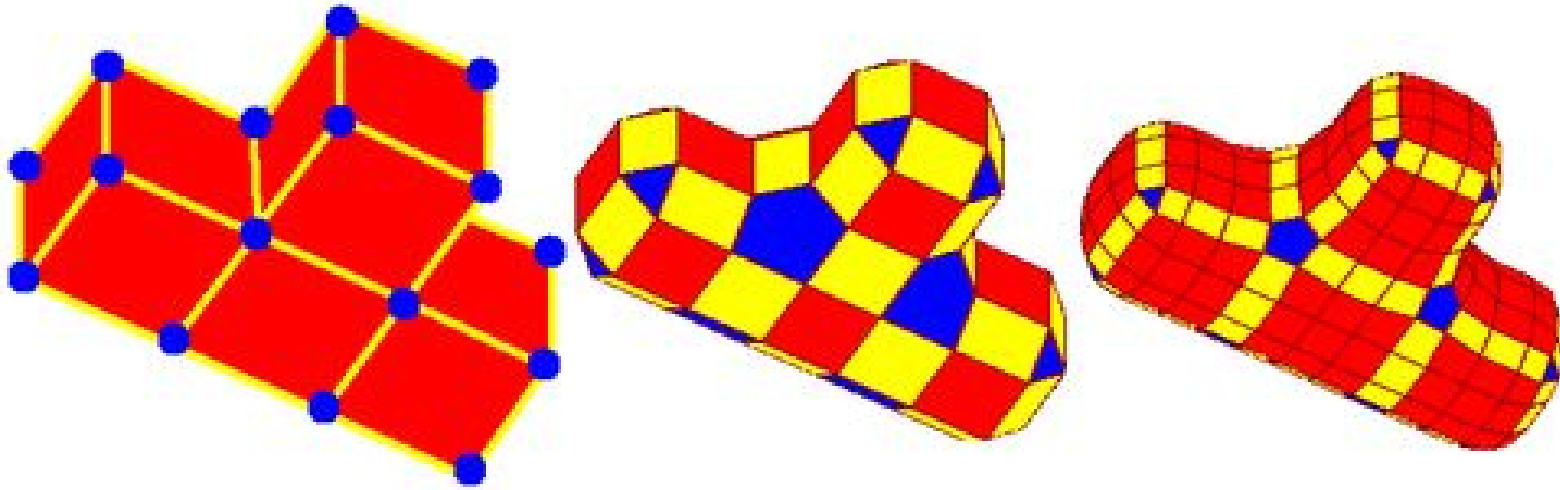
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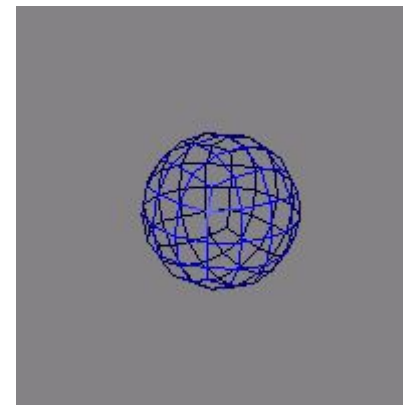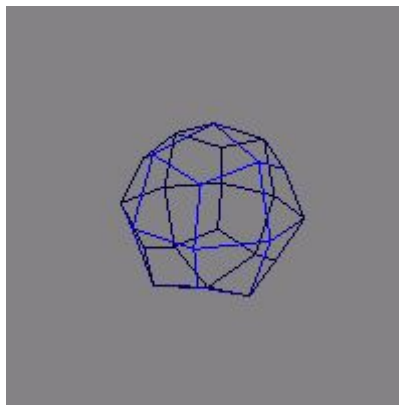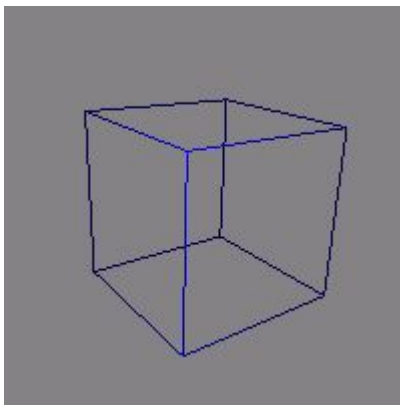
Doo-Sabin Subdivision

- Proposed by Doo and Sabin in 1978
- An extension of Chaiken's algorithm to 3D mesh surfaces

http://www.rose-hulman.edu/~finn/CCLI/Applets/DooSabinApplet.html

Catmull-Clark Subdivision

- A face with n edges are subdivided into n quadrilaterals
- Quads are better than triangles at capturing the symmetries of natural and man-made objects.
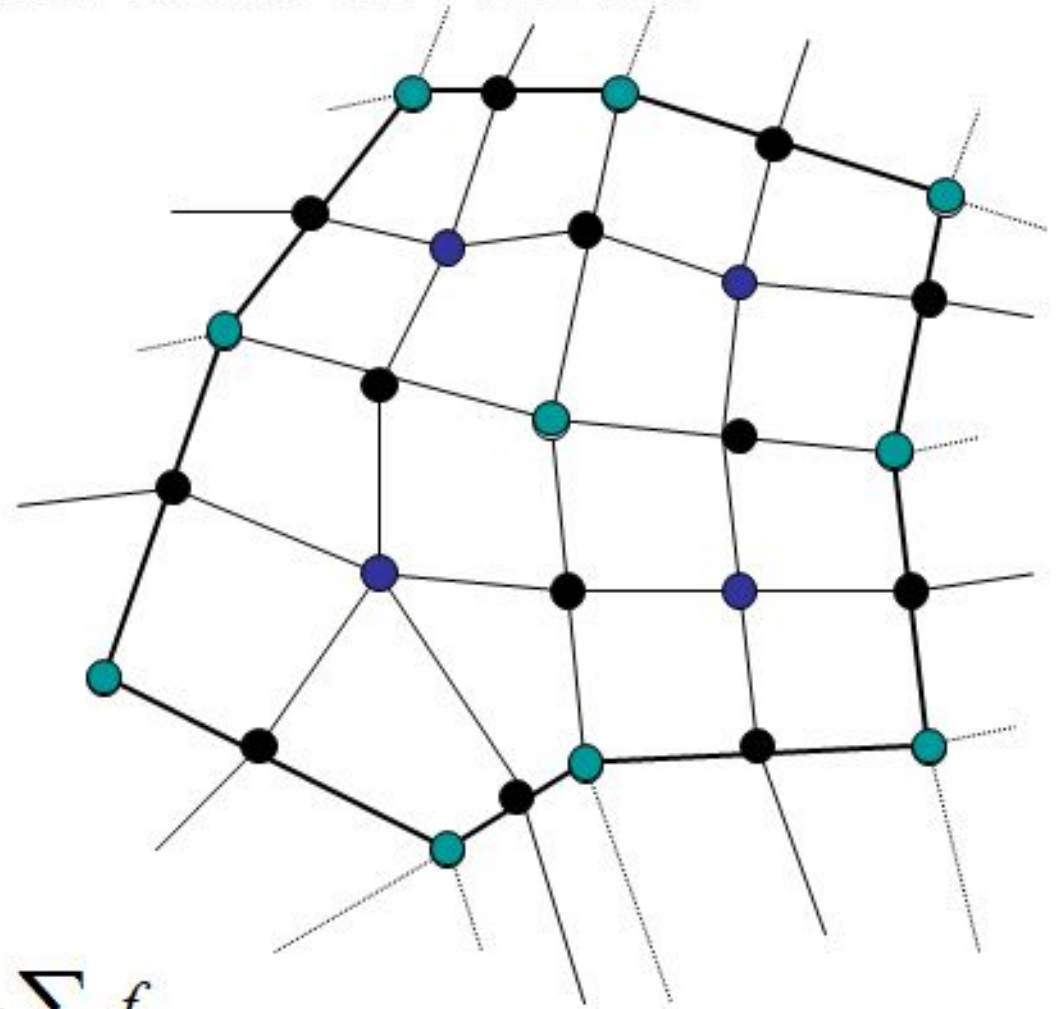
Catmull-Clark Subdivision

- FACE

$$f = \frac{1}{n}\sum_{1}^{n} v_i$$

- EDGE

$$e = \frac{v_1 + v_2 + f_1 + f_2}{4}$$

○ → ● VERTEX

$$v_{i+1} = \frac{n-2}{n}v_i + \frac{1}{n^2}\sum_{j} e_j + \frac{1}{n^2}\sum_{j} f_j$$

http://www.rose-hulman.edu/~finn/CCLI/Applets
/CatmullClarkApplet.html

Modeling with Catmull-Clark

- Subdivision produces smooth continuous surfaces.
- How can "sharpness" and creases be controlled in a modeling environment?

ANSWER: Define new subdivision rules for "creased" edges and vertices.

- Tag Edges sharp edges.

- If an edge is sharp, apply new sharp subdivision rules.

- Otherwise subdivide with normal rules.

Sharp Edges…

- Tag Edges as "**sharp**" or "not-sharp"

  - n = 0 – "**not sharp**"

  - n > 0 – sharp
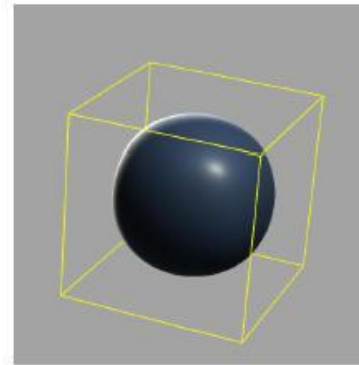
During Subdivision,

- if an edge is "sharp", use sharp subdivision rules. Newly created edges, are assigned a sharpness of n-1.

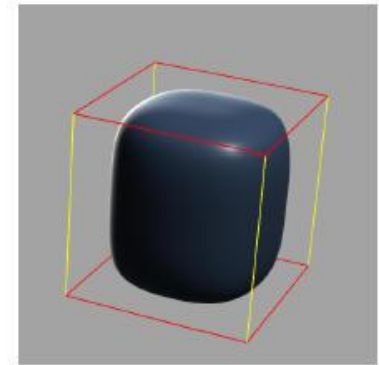- If an edge is "not-sharp", use normal smooth subdivision rules.

  IDEA: Edges with a sharpness of "n" do not get subdivided smoothly for "n" iterations of the algorithm.

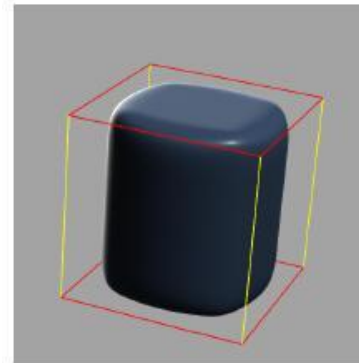- In the picture on the right, the control mesh is a unit cube
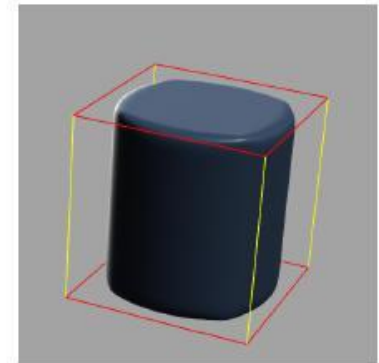
- Different sharpness applied

(a)

(b)

(c)

(d)
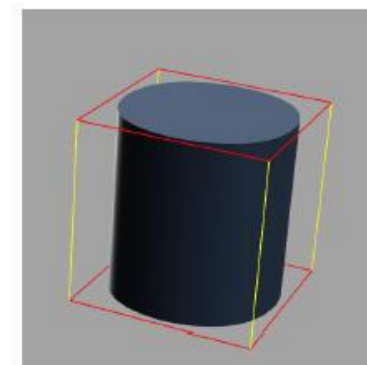
(e)

Sharp Rules

FACE (unchanged)

$$f = \frac{1}{n} \sum_{1}^{n} v_i$$

EDGE

$$e = \frac{v_1 + v_2}{2}$$
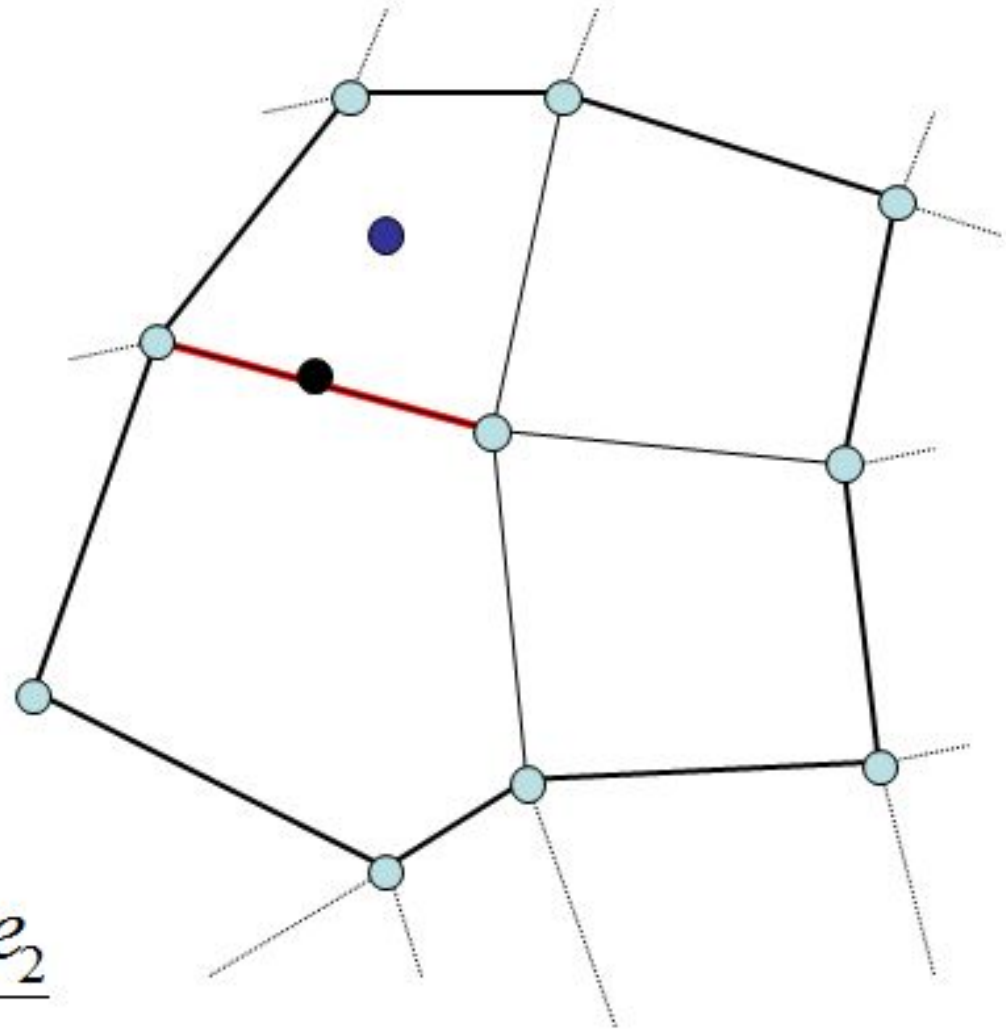
VERTEX

adj. Sharp edges

| corner | >2 |
| crease | 2 |

$$v_{i+1} = v_i$$

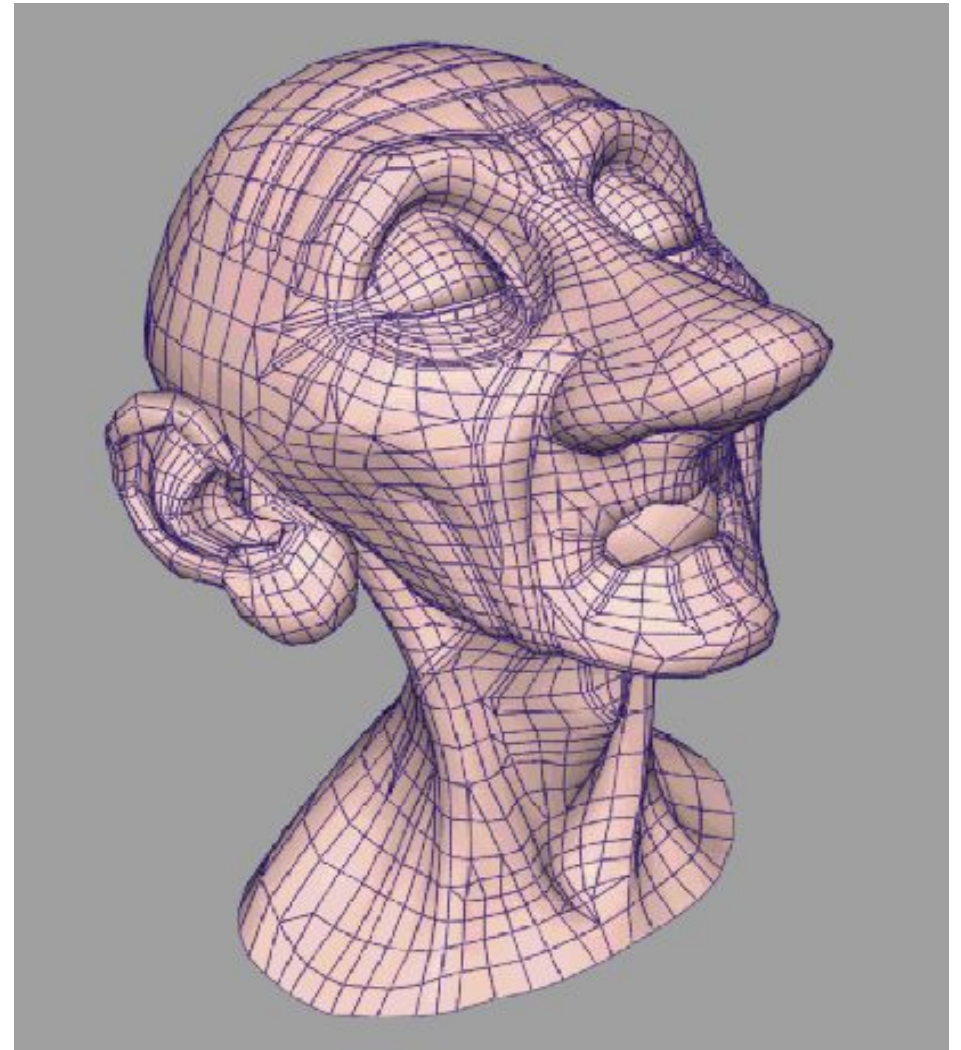$$v_{i+1} = \frac{e_1 + 6v_i + e_2}{8}$$

Another example of creases

Subdivision Surfaces in character animation [DeRose98]

- Used for first time in Geri's game to overcome topological restriction of NURBS

- Modeled Geri's head, hands, jacket, pants, shirt, tie, and shoes

Geri's Game

- Academy Award winning movie by Pixar
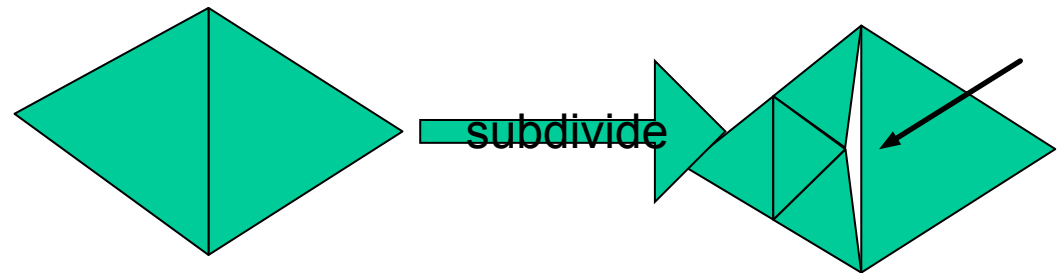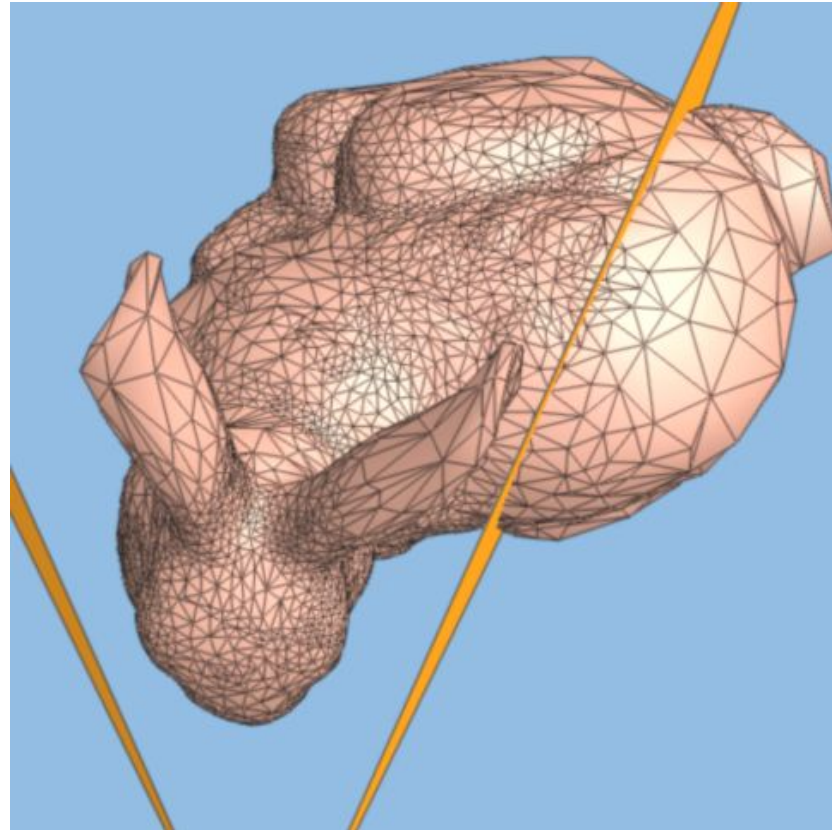 - Made by subdivision surface

[http://www.youtube.com/watch?v=1m7dcbIKvlw]

Demo of Catmull-Clark subdivision surface

- http://www.youtube.com/watch?v=lU8f0hnorU8&feature=related

Adaptive Subdivision

- Not all regions of a model need to be subdivided.

- Idea: Use some criteria and adaptively subdivide mesh where needed.
  - Curvature
  - Screen size (make triangles < size of pixel)
  - View dependence
    - Distance from viewer
    - Silhouettes
    - In view frustum
  - Careful! Must ensure that "cracks" aren't made

subdivide

Subdivision Surface Summary

- Advantages
  - Simple method for describing complex surfaces
  - Relatively easy to implement
  - Arbitrary topology
  - Local support
  - Guaranteed continuity
  - Multi-resolution
- Difficulties
  - Intuitive specification
  - Parameterization
  - Intersections

- Edwin Catmull

  Utah – NYIT – Lucas Films – Pixar -

  present: President of Disney Animation
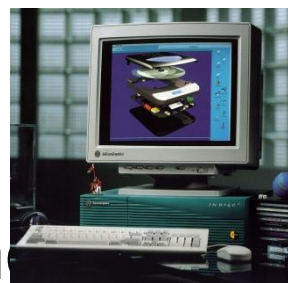   Studios and Pixar Animation Studios

- Jim Clark

  Utah – UCSC - Stanford – Silicon Graphics -
    Netscape - …

  He also co-produced the movie _____ Cove.

Readings

- Shirley - 15.6.3, 15.6.4
- Akenine-Möller - 13.5.4 (Catmull-Clark subdivision)
- A very good website for parametric curves / surfaces
  http://www.cs.mtu.edu/~shene/COURSES/cs3621/

- DeRose, Tony, Michael Kass, and Tien Truong. Subdivision Surfaces in Character Animation. *SIGGRAPH 98*.

- Clark, E., and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Geometric Design*, Vol. 10, No. 6, 1978.

- Doo, D. and M. Sabin. Behavior of Recursive Division Surfaces Near Extraordinary Points. Computer-Aided Design. Vol. 10, No. 6, 1978.