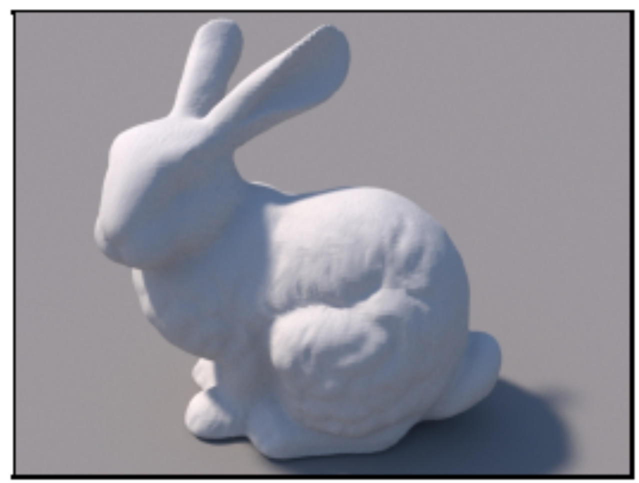
Computer Graphics

Lecture 16

Curves and Surfaces I

Characters and Objects

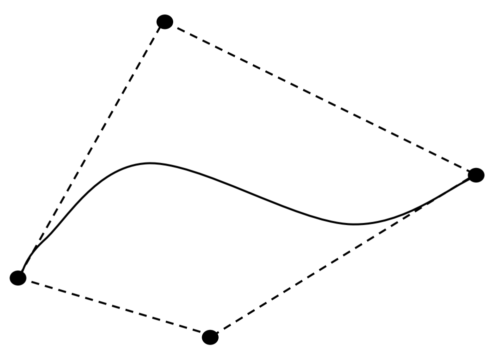
- Important for composing the scene
- Need to design and model them in the first place

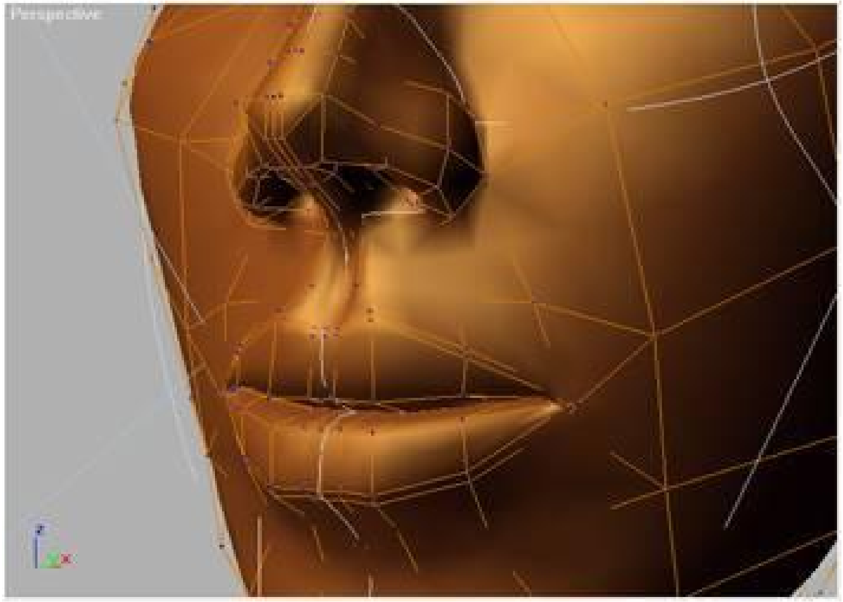


Curves / curved surfaces

Can produce smooth surfaces with less parameters

- Easier to design
- Can efficiently preserve complex structures

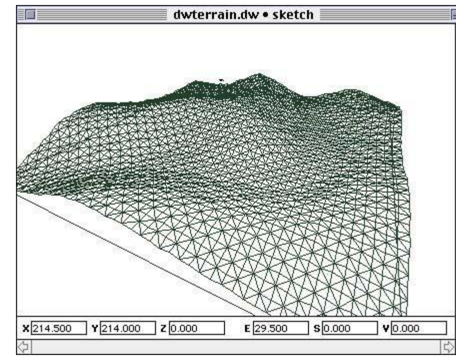




Today

- Parametric curves
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- Bicubic patches
- Tessellation
  - Adaptive tesselation

Types of Curves and Surfaces

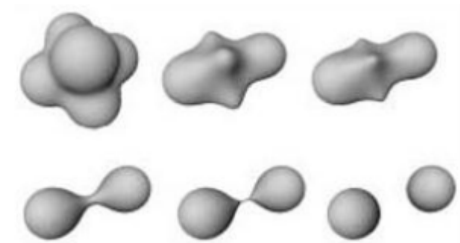


- Explicit:

$$y = mx + b \qquad\qquad r = A_r x + B_r y + C_r$$

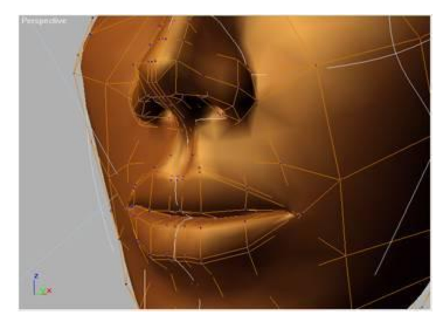
- Implicit:

$$Ax + By + C = 0 \qquad\qquad (x - x_0)^2 + (y - y_0)^2 - r^2 = 0$$



- Parametric:

$$x = x_0 + (x_1 - x_0)t \qquad\qquad x = x_0 + r\cos\theta$$
$$y = y_0 + (y_1 - y_0)t \qquad\qquad y = y_0 + r\sin\theta$$



Why parametric?

- Simple and flexible
- The function of each coordinates can be defined independently.

  (x(t), y(t)) : 1D curve in 2D space

  (x(t), y(t), z(t)) : 1D curve in 3D space

  (x(s,t), y(s,t), z(s,t)) : 2D surface in 3D space
- Polynomial are suitable for creating smooth surfaces with less computation

$$x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

Today

- Parametric curves
  - Introduction
  - **Hermite curves**
  - **Bezier curves**
  - **Uniform cubic B-splines**
  - **Catmull-Rom spline**
- Bicubic patches
- Tessellation
  - Adaptive tesselation

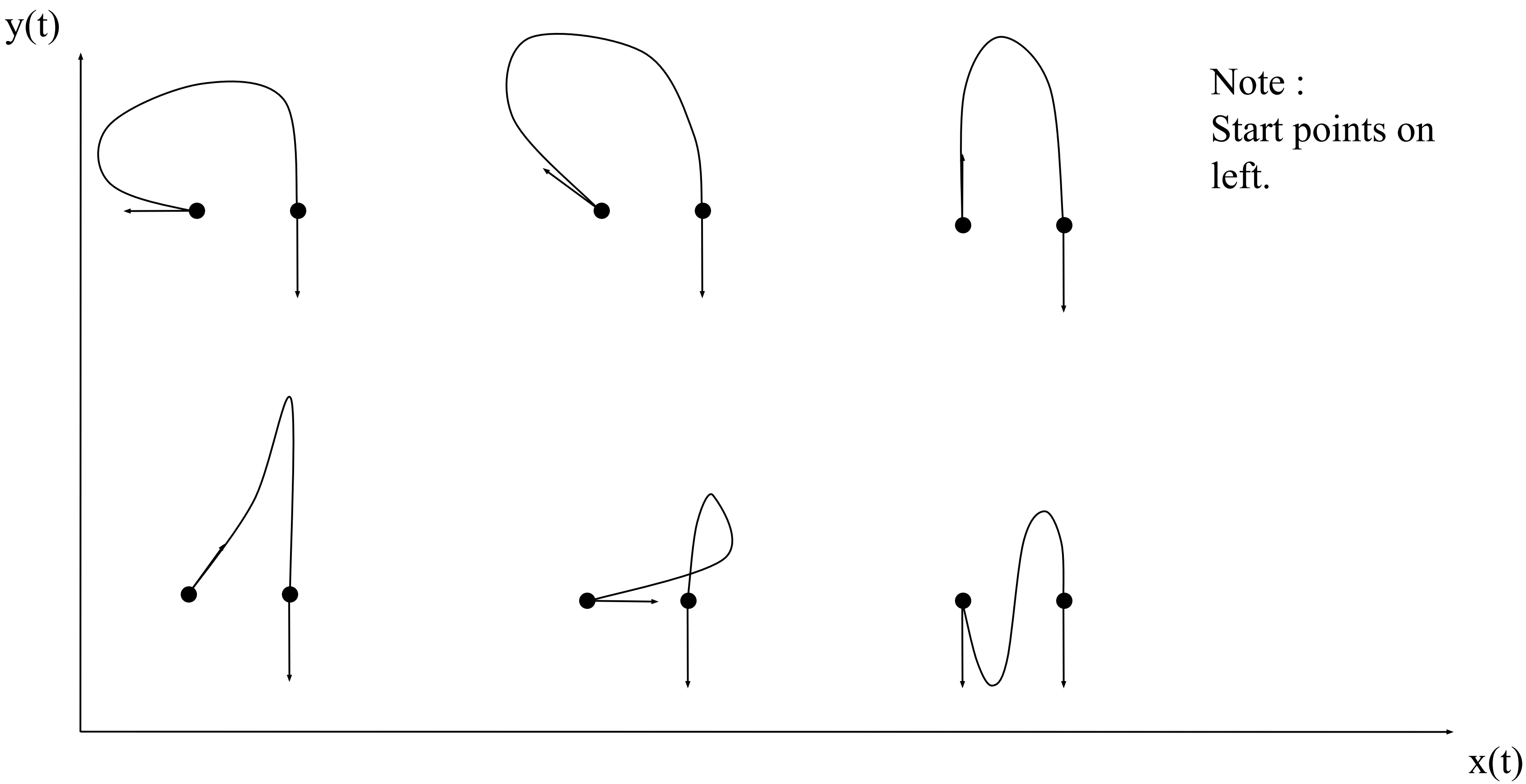
Hermite curves

Hermite Specification

- A cubic polynomial

  $x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$
- *t* ranging from 0 to 1
- Polynomial can be specified by the position of, and gradient at, each endpoint of curve.

Family of Hermite curves.



Note :
Start points on left.

http://www.rose-hulman.edu/~finn/CCLI/Applets/CubicHermiteApplet.html

Finding Hermite coefficients

Can solve them by using the boundary conditions

$\boxed{X(t) = a_3t^3 + a_2t^2 + a_1t + a_{0,} \quad X'(t) = 3a_3t^2 + 2a_2t + a_1}$

Equation and derivative
Want to compute ai

Substituting for t at each endpoint:

$x_0 = X(0) = a_0$ $\qquad$ $x_0' = X'(0) = a_1$

$x_1 = X(1) = a_3 + a_2 + a_1 + a_0$ $\qquad$ $x_1' = X'(1) = 3a_3 + 2a_2 + a_1$

And the solution is:

$a_0 = x_0$ $\qquad$ $a_1 = x_0'$

$a_2 = -3x_0 – 2x_0' + 3x_1 – x_1'$ $\qquad$ $a_3 = 2x_0 + x_0' - 2x_1 + x_1'$

$X(t) = (2x_0 + x_0' - 2x_1 + x_1')\, t^3 + (-3x_0 – 2x_0' + 3x_1 – x_1')\, t^2 + (x_0')\, t + x$

Finding Hermite coefficients

Can solve them by using the boundary conditions

$X(t) = a_3t^3 + a_2t^2 + a_1t + a_{0,}$ $\quad X'(t) = 3a_3t^2 + 2a_2t + a_1$

Substituting for t at each endpoint:

$x_0 = X(0) = a_0$ $\qquad x_0' = X'(0) = a_1$

$x_1 = X(1) = a_3 + a_2 + a_1 + a_0$ $\qquad x_1' = X'(1) = 3a_3 + 2a_2 + a_1$

boundary conditions

And the solution is:

$a_0 = x_0$ $\qquad a_1 = x_0'$

$a_2 = -3x_0 – 2x_0' + 3x_1 – x_1'$ $\qquad a_3 = 2x_0 + x_0' - 2x_1 + x_1'$

$X(t) = (2x_0 + x_0' - 2x_1 + x_1')\, t^3 + (-3x_0 – 2x_0' + 3x_1 – x_1')\, t^2 + (x_0')\, t +$
x

Finding Hermite coefficients

Can solve them by using the boundary conditions

$X(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_{0,} \quad X'(t) = 3a_3 t^2 + 2a_2 t + a_1$

Substituting for t at each endpoint:

$x_0 = X(0) = a_0 \qquad x_0' = X'(0) = a_1$

$x_1 = X(1) = a_3 + a_2 + a_1 + a_0 \qquad x_1' = X'(1) = 3a_3 + 2a_2 + a_1$

And the solution is:

solving for coefficients

$a_0 = x_0 \qquad a_1 = x_0'$

$a_2 = -3x_0 – 2x_0' + 3x_1 – x_1' \qquad a_3 = 2x_0 + x_0' - 2x_1 + x_1'$

$X(t) = (2x_0 + x_0' - 2x_1 + x_1')\, t^3 + (-3x_0 – 2x_0' + 3x_1 – x_1')\, t^2 + (x_0')\, t +$

x

The Hermite matrix: M_H

The resultant polynomial can be expressed in matrix form:

$$X(t) = t^T M_H q \qquad \text{(q is the control vector)}$$

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{bmatrix}$$

We can now define a parametric polynomial for each coordinate required independently, ie. X(t), Y(t) and Z(t)

Hermite Basis (Blending) Functions

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ -3 & -2 & 3 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{bmatrix}$$

$$= \underline{\underline{(2t^3 - 3t^2 + 1)}}x_0 + \underline{\underline{(t^3 - 2t^2 + t)}}x_0' + \underline{\underline{(-2t^3 + 3t^2)}}x_1 + \underline{\underline{(t^3 - t^2)}}x_1'$$

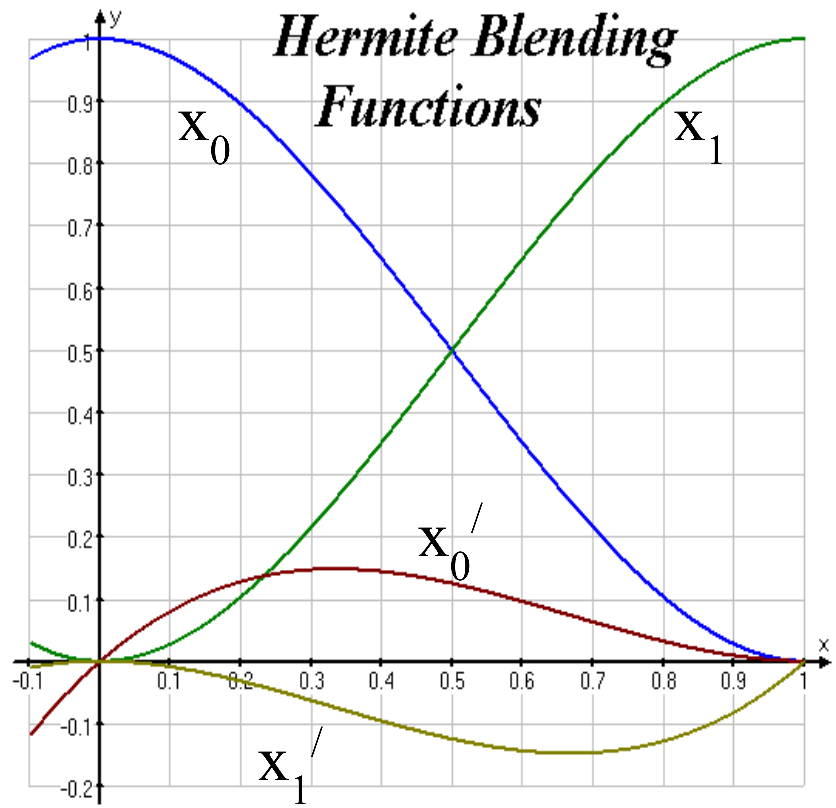
Hermite Basis (Blending) Functions

$$X(t) = \underline{\underline{(2t^3 - 3t^2 + 1)}}x_0 + \underline{\underline{(t^3 - 2t^2 + t)}}x_0' + \underline{\underline{(-2t^3 + 3t^2)}}x_1 + \underline{\underline{(t^3 - t^2)}}x_1'$$

The graph shows the shape of the four basis functions – often called *blending functions*.

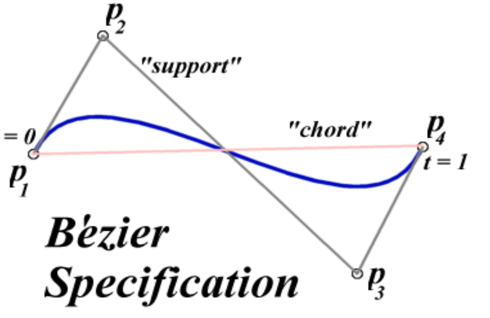
They are labelled with the elements of the control vector that they weight.

Note that at each end only position is non-zero, so the curve must touch the endpoints



Bézier Curves

- Hermite cubic curves are difficult to model – need to specify point and gradient.
- Paul de Casteljau who was working for Citroën, invented another way to compute the curves
- Publicised by Pierre Bézier from Renault
- By only giving points instead of the derivatives



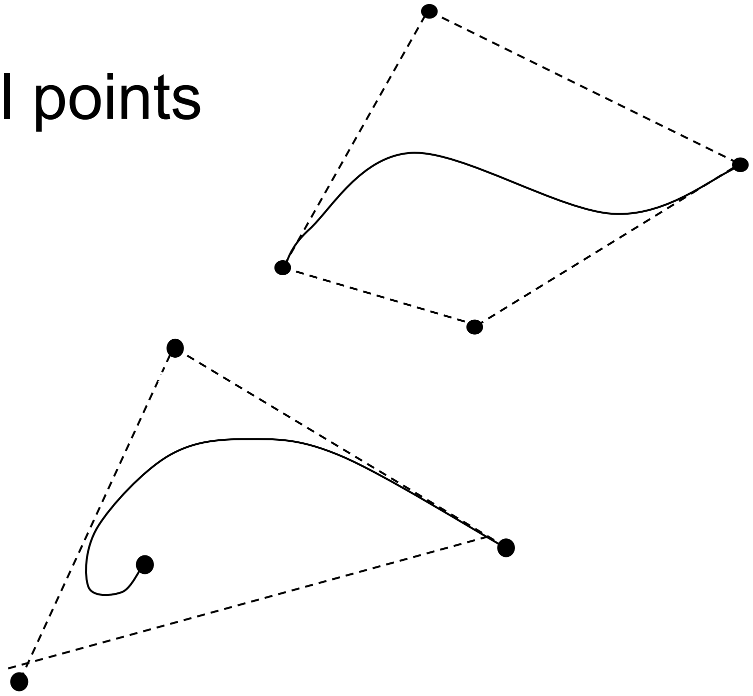
Bézier Curves (2)

Can define a curve by specifying 2 endpoints and 2 additional control points

The two middle points are used to specify the gradient at the endpoints

Fit within the convex hull by the control points

http://www.rose-hulman.edu/~finn/CCLI/Applets/BezierBernsteinApplet.html



Bézier Matrix

- The cubic form is the most popular
  $X(t) = t^T M_B q$ (M_B is the Bézier matrix)
- With *n=4* and *r=0,1,2,3* *we get:*

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

- Simil

$$X(t) = (-t^3 + 3t^2 - 3t + 1)q_0 + (3t^3 - 6t^2 + 3t)q_1 + (-3t^3 + 3t^2)q_2 + (t^3)q_3$$

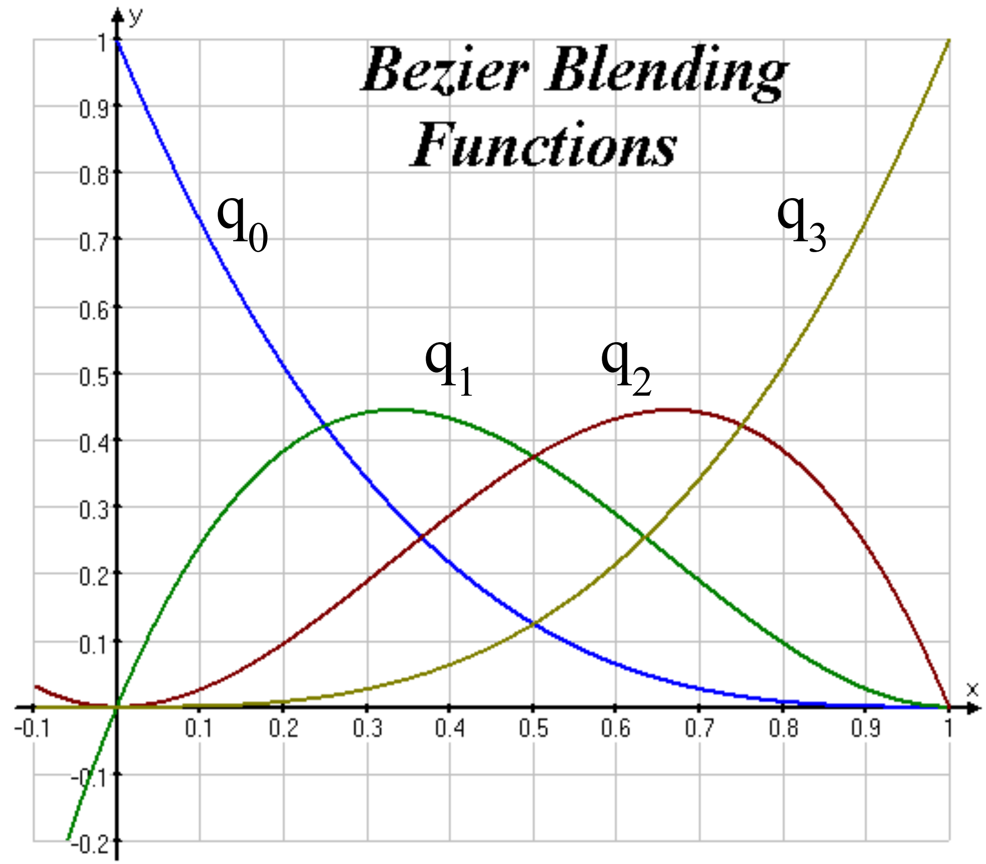
Bézier blending functions

This is how the polynomials for each coefficient looks like

The functions sum to 1 at any point along the curve.

Endpoints have full weight

The weights of each function is clear and the labels show the control

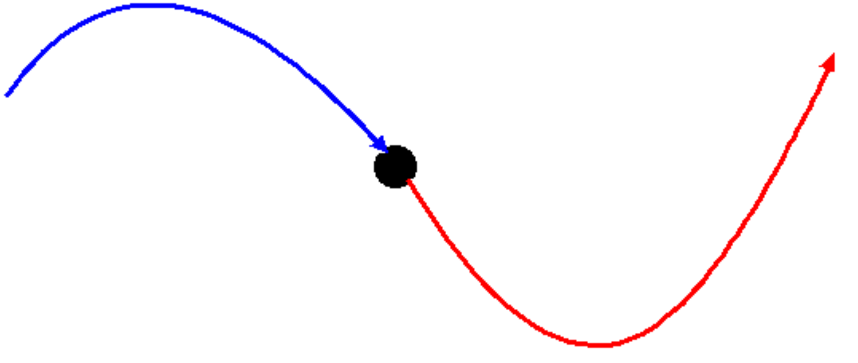


How to produce complex, long curves?

- We could only use 4 control points to design curves.
- What if we want to produce long curves with complex shapes.
- How do can we do that?

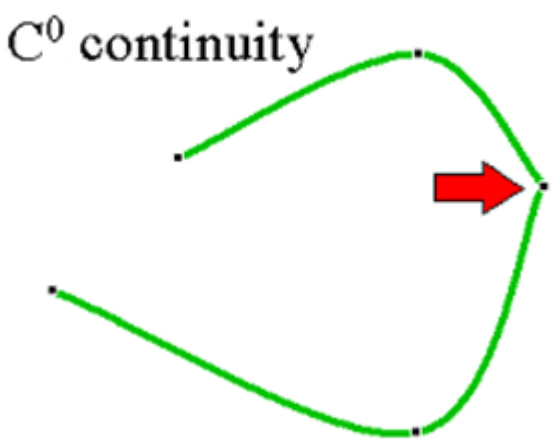
Drawing Complex Long Curves

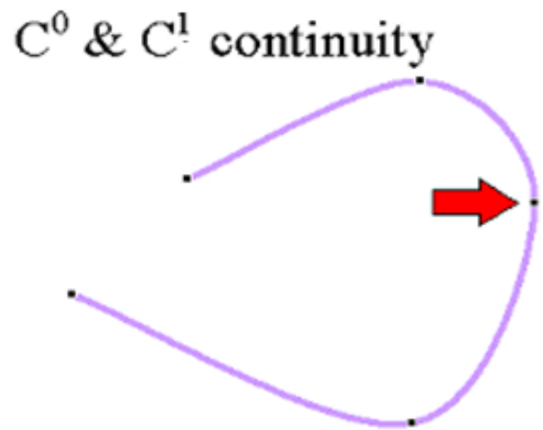
- Using higher order curves
  - costly
  - Need many multiplications
- Pierce together low order curves
  - Need to make sure the connection points are smooth



Continuity between curve segments

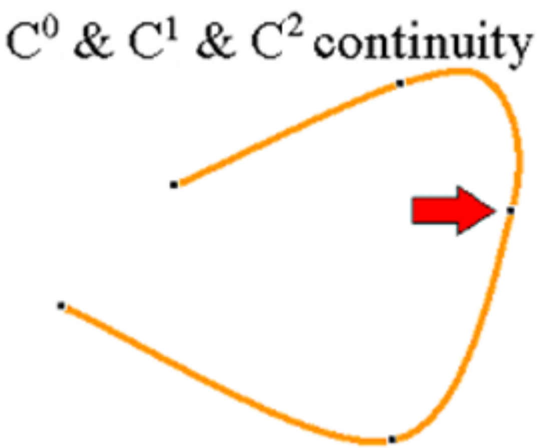
- If the direction and magnitude of $d\ / dt^n\ [X(t)]$ are equal at the join point, the curve is called $\boldsymbol{C^n}$ **continuous**
- i.e. if two curve segments are simply connected, the curve is $\boldsymbol{C_0}$ **continuous**
- If the tangent vectors of two cubic curve segments are equal at the join point, the curve is $\boldsymbol{C_1}$ ***continuous***

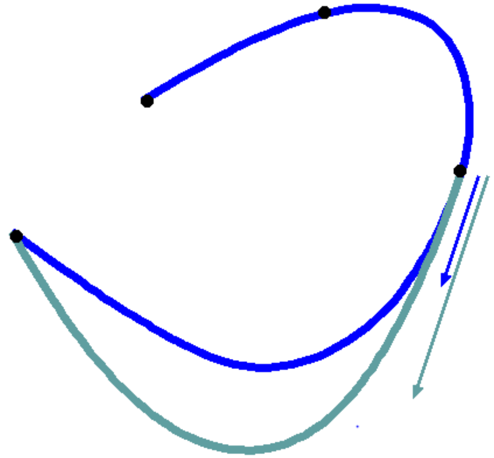




Continuity between curve segments

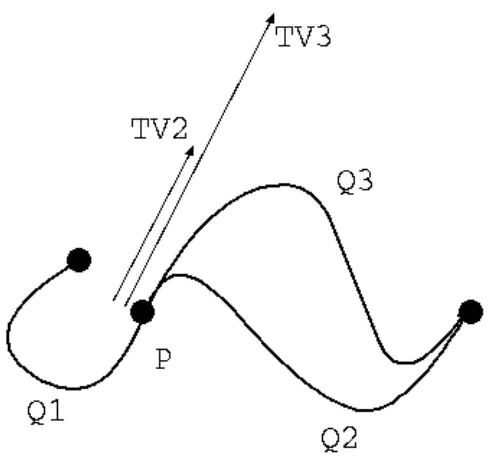
- If the directions (but not necessarily the magnitudes) of two segments' tangent vectors are equal at the join point, the curve has $\boldsymbol{G^1}$ **continuity**

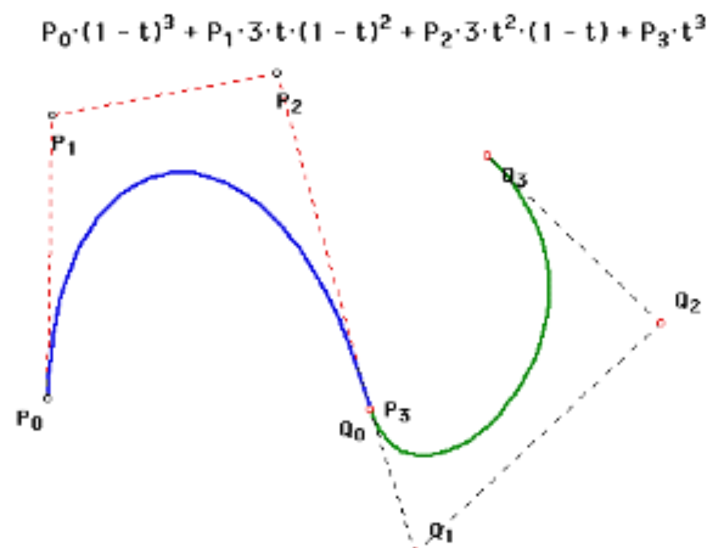




Continuity with Hermite and Bezier Curves

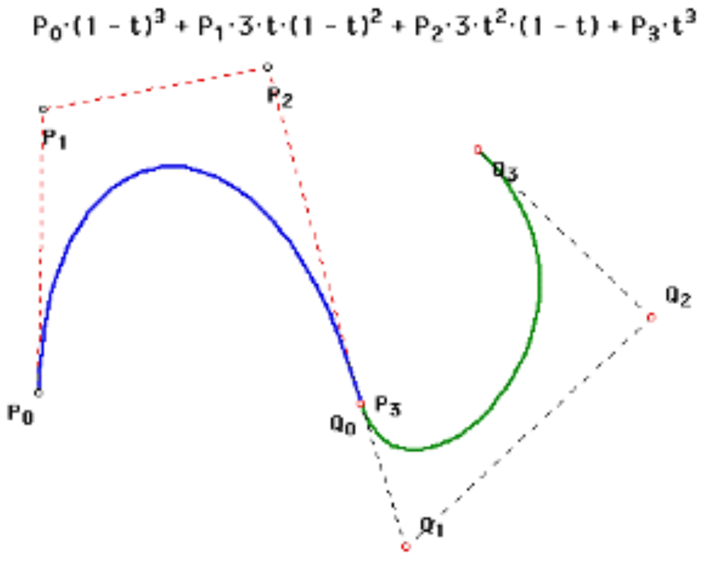
– How to achieve C0,C1,G1 continuity?





Joining Bezier Curves

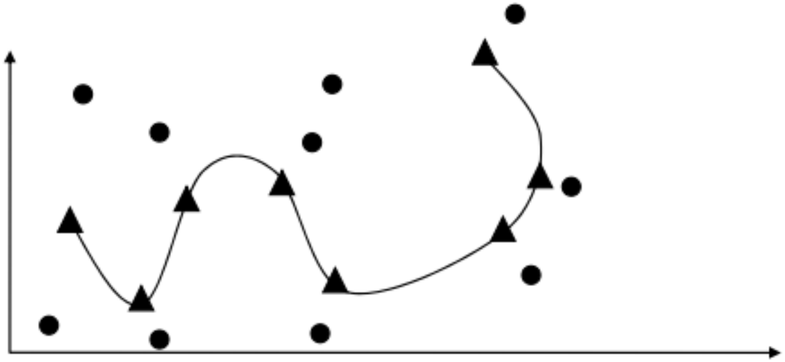
- G^1 continuity is provided at the endpoint when $P_3 - P_4 = k\,(P_4 - P_5)$
- if $k=1$, C^1 continuity is obtained



Uniform Cubic B-Splines

- Another popular form of curves
- The curve does not necessarily pass through the control points
- Can produce a longer continuous curve without worrying about the boundaries
- Has C_2 continuity at the boundaries

http://www.rose-hulman.edu/~finn/CCLI/Applets/BSplineApplet.html



Uniform Cubic B-Splines (2)

- The matrix form and the basis functions
- The knots specify the range of the curve

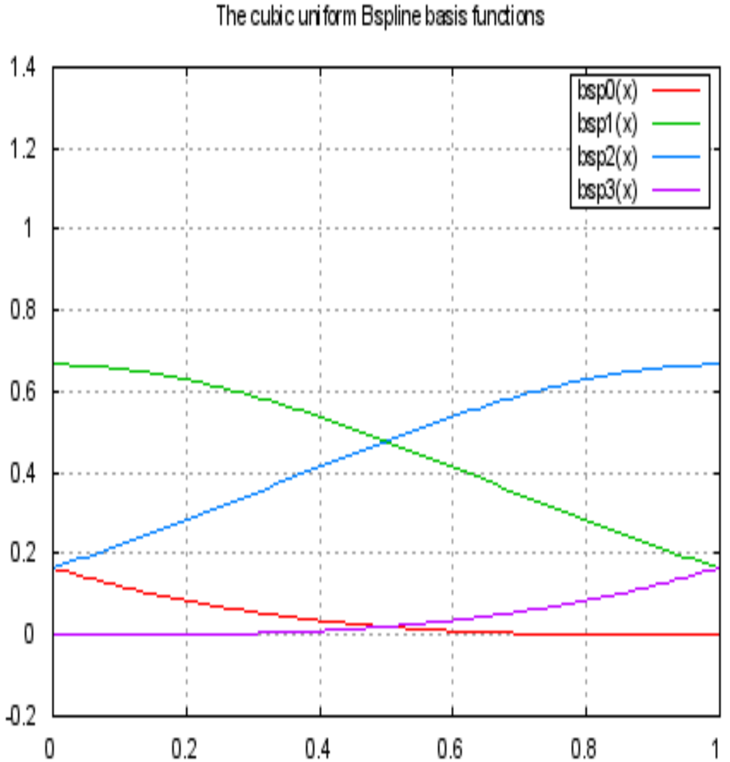
$$X(t) = \mathbf{t}^{\mathrm{T}}\mathbf{M}\mathbf{Q}^{(\mathbf{i})} \qquad for \quad t_i \le t \le t_{i+1}$$

$$where \qquad \mathbf{Q}^{(\mathbf{i})} = (x_{i-3}, \ldots, x_i)$$

$$\mathbf{M} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix},$$

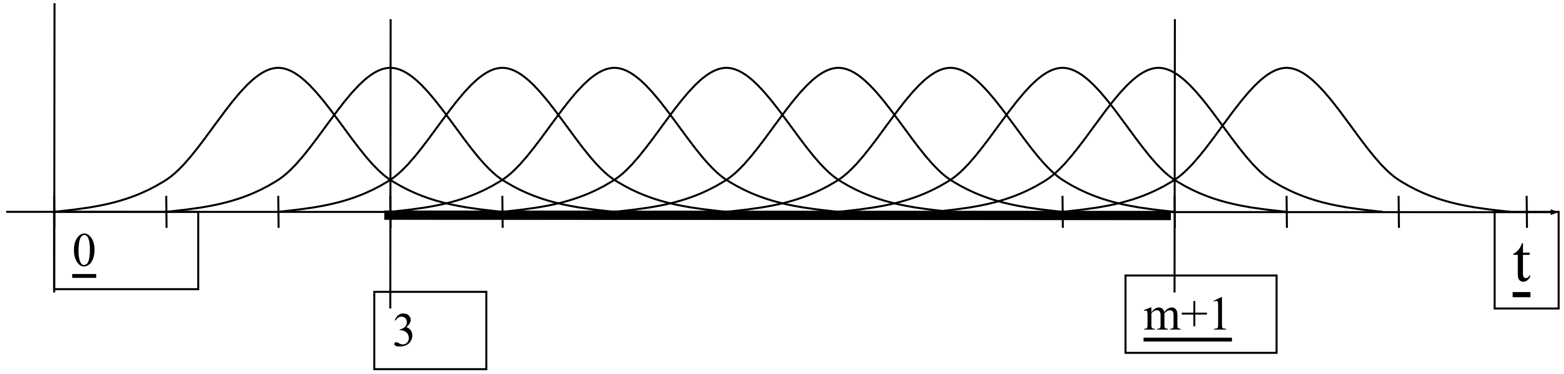
$$\mathbf{t}^{\mathrm{T}} = \left((t-t_i)^3, (t-t_i)^2, t-t_i, 1\right)$$

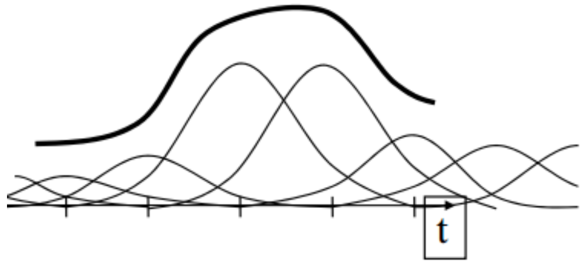
$$t_i : \text{knots}, \quad 3 \le i$$



Uniform Cubic B-Splines (3)

- This is how the basis look like over the domain
- The initial part is defined after passing the fourth knot





Another usage of uniform cubic B-splines

- Representing the joint angle trajectories of characters and robots
- Need more control points to represent a longer continuous movement
- Need C_2 continuity to make the acceleration smooth
- And not changing the torques suddenly

Catmull-Rom Spline

- A curve that interpolates control points
- The tangent vectors at the endpoints of a Hermite curve is set such that they are decided by the two surrounding control points

Hermite Specification

Catmull-Rom Spline

- C_1 continuity

$$P^i(t) = T \cdot M_{CR} \cdot G_B$$

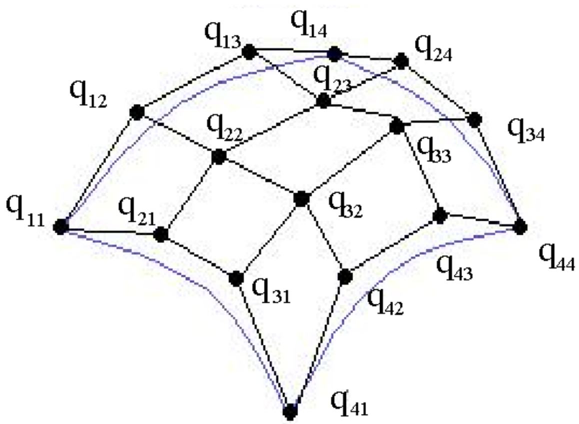
$$= \frac{1}{2} \cdot T \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

Today

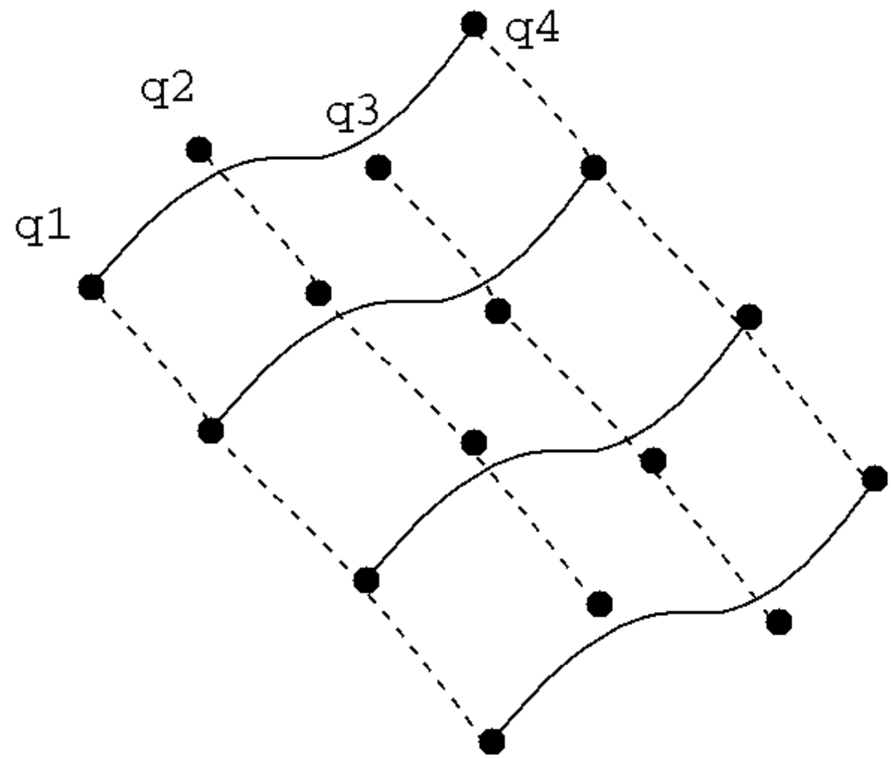
- Parametric curves
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- **Bicubic patches**
- Tessellation
  - Adaptive tessellation

Bicubic patches

- The concept of parametric curves can be extended to surfaces
- The cubic parametric curve is in the form of $Q(t) = \boldsymbol{t}^T \boldsymbol{M}\, \boldsymbol{q}$ where $\boldsymbol{q}=(q_1,q_2,q_3,q_4)$: q_i control points, $\boldsymbol{M}$ is the basis matrix (Hermite or Bezier,…), $\boldsymbol{t}^T = (t^3, t^2, t, 1)$



- Now we assume q_i to vary along a parameter s,
- $Q_i(s,t)=\mathbf{t}^T\mathbf{M}\ [q_1(s),q_2(s),q_3(s),q_4(s)]$
- $q_i(s)$ are themselves cubic curves, we can write them in the form …



Bicubic patches

$$Q(s,t) = t^T.M.(s^T.M.[\mathbf{q}_{11},\mathbf{q}_{12},\mathbf{q}_{13},\mathbf{q}_{14}],\ldots,s^T.M.[\mathbf{q}_{41},\mathbf{q}_{42},\mathbf{q}_{43},\mathbf{q}_{44}$$

$$= t^T.M.\mathbf{q}.M^T.s$$

where **q** is a 4x4 matrix $\begin{bmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ q_{12} & q_{22} & q_{32} & q_{42} \\ q_{13} & q_{23} & q_{33} & q_{43} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix}$

Each column contains the control points of

$q_1(s),\ldots,q_4(s)$

x,y,z computed by

$$x(s,t) = t^T.M.\mathbf{q}_x.M^T.s$$

$$y(s,t) = t^T.M.\mathbf{q}_y.M^T.s$$

$$z(s,t) = t^T.M.\mathbf{q}_z.M^T.s$$

Bézier example

- We compute (x,y,z) by

$$x(s,t) = t^T.M_B.q_x.M_B^T.s$$

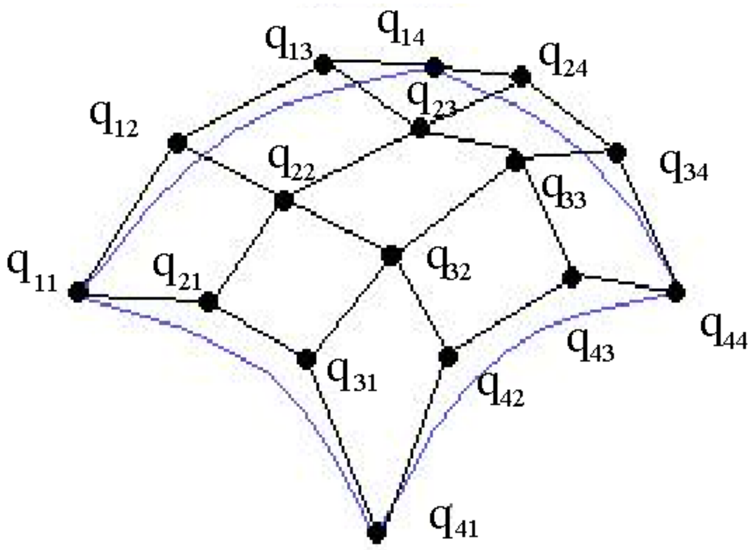
$$q_x \text{ is } 4\times 4 \text{ array of } x \text{ coords}$$

$$y(s,t) = t^T.M_B.q_y.M_B^T.s$$

$$q_y \text{ is } 4\times 4 \text{ array of } y \text{ coords}$$

$$z(s,t) = t^T.M_B.q_z.M_B^T.s$$

$$q_z \text{ is } 4\times 4 \text{ array of } z \text{ coords}$$



http://www.math.psu.edu/dlittle/java/parametricequations/beziersurfaces/index.html

Today

- Parametric curves
  - Introduction
  - Hermite curves
  - Bezier curves
  - Uniform cubic B-splines
  - Catmull-Rom spline
- Bicubic patches
- **Tessellation**
  - Adaptive tessellation

Displaying Bicubic patches.

- Directly rasterizing bicubic patches is not so easy
- Need to convert the bicubic patches into a polygon mesh
  - tessellation
- Need to compute the normals
  - vector cross product of the 2 tangent vectors.

Normal Vectors

$$\frac{\partial}{\partial s} Q(s,t) = \frac{\partial}{\partial s}(t^T.M.q.M^T.s) = t^T.M.q.M^T.\frac{\partial}{\partial s}(s)$$

$$= t^T.M.q_x.M^T.[3s^2,2s,1,0]^T$$

$$\frac{\partial}{\partial t} Q(s,t) = \frac{\partial}{\partial t}(t^T.M.q.M^T.s) = \frac{\partial}{\partial t}(t^T).M.q.M^T.s$$

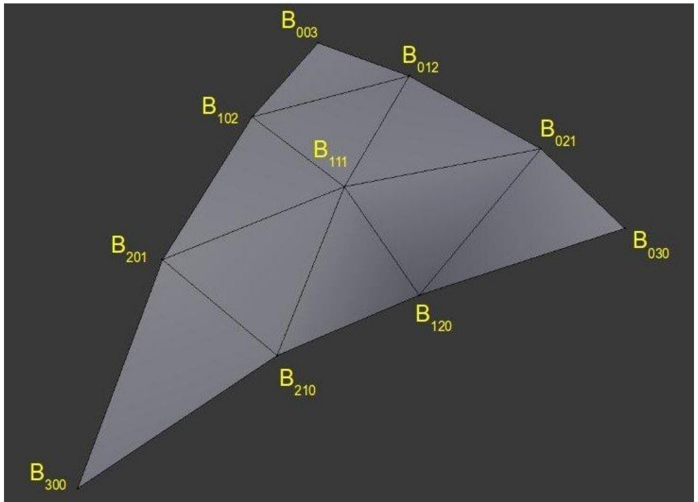
$$= [3t^2,2t,1,0]^T.M.q.M^T.s$$

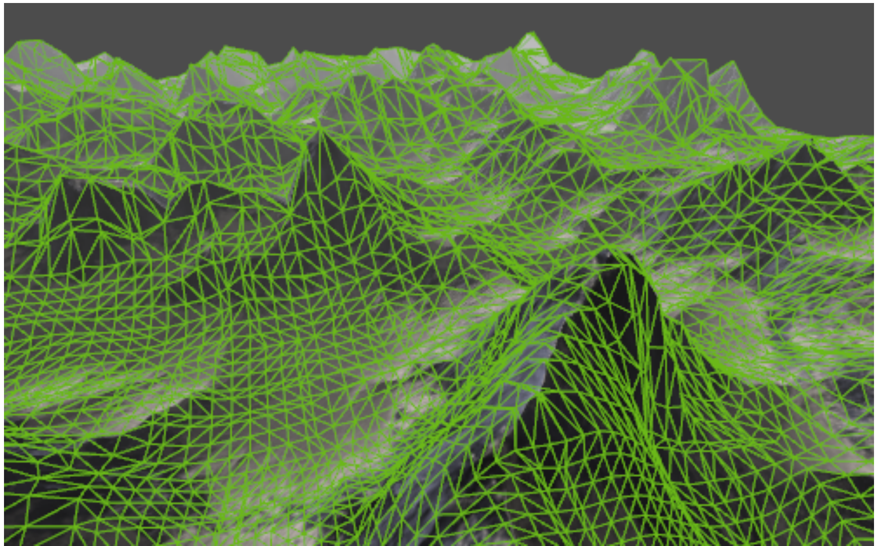
$$\frac{\partial}{\partial s} Q(s,t) \times \frac{\partial}{\partial t} Q(s,t) = (y_s z_t - y_t z_s, z_s x_t - z_t x_s, x_s y_t - x_t y_s)$$

Tangent vectors can be computed by computing the partial derivatives
Then computing the cross product of the two partial derivative vectors

Tessellation

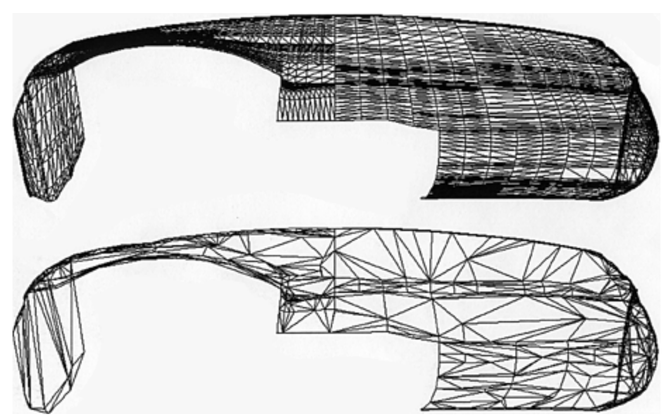
- As computers are optimized for rendering triangles, the easiest way to display parametric surfaces is to convert them into triangle meshes
- The simplest way is to do uniform tessellation, which samples points uniformly in the parameter space





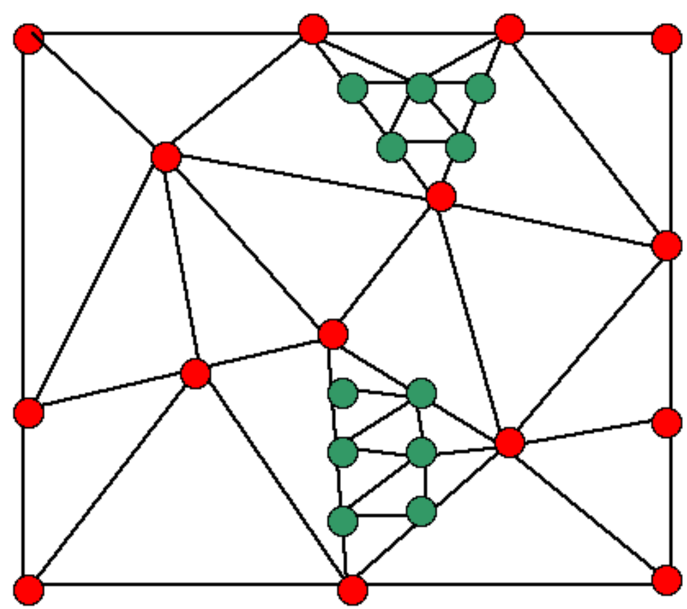
Uniform Tessellation

- Sampling points uniformly with the parameters
- What are the problems with uniform tessellation?
- Which area needs more tessellation?
- Which area does not need much tessellation?



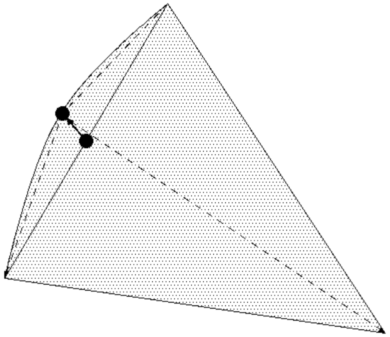
Adaptive Tessellation

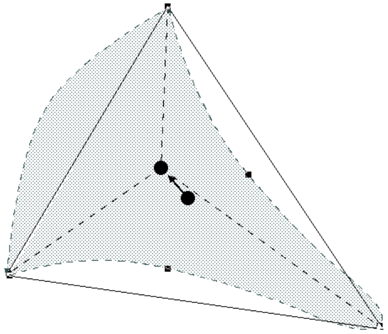
- Adaptive tessellation – adapt the size of triangles to the shape of the surface



Adaptive Tessellation

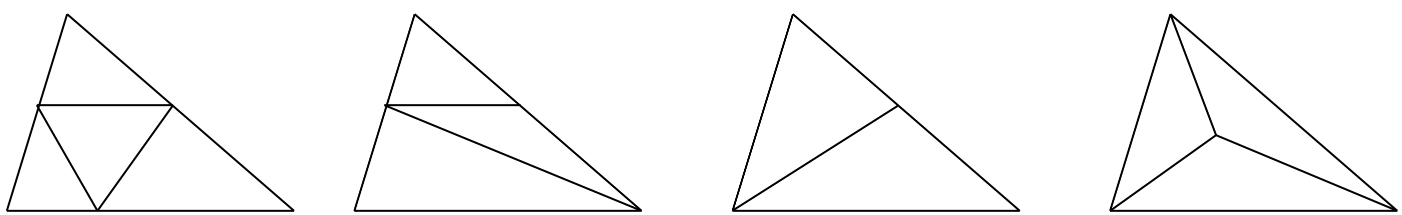
- For every triangle edges, check if each edge is tessellated enough (curveTessEnough())
- If all edges are tessellated enough, check if the whole triangle is tessellated enough as a whole (triTessEnough())
- If one or more of the edges or the triangle's interior is not tessellated enough, then further tessellation is needed





Adaptive Tessellation

- When an edge is not tessellated enough, a point is created halfway between the edge points' uv-values
- New triangles are created and the tessellator is once again called with the new triangles as input



Four cases of further tessellation

Adaptive Tessellation

AdaptiveTessellate(p,q,r)

- tessPQ=not curveTessEnough(p,q)
- tessQR=not curveTessEnough(q,r)
- tessRP=not curveTessEnough(r,p)
- If (tessPQ and tessQR and tessRP)
  - AdaptiveTessellate(p,(p+q)/2,(p+r)/2);
  - AdaptiveTessellate(q,(q+r)/2,(q+p)/2);
  - AdaptiveTessellate(r,(r+p)/2,(r+q)/2);
  - AdaptiveTessellate((p+q)/2,(q+r)/2,(r+p)/2);
- else if (tessPQ and tessQR)
  - AdaptiveTessellate(p,(p+q)/2,r);
  - AdaptiveTessellate((p+q)/2,(q+r)/2,r);
  - AdaptiveTessellate((p+q)/2,q,(q+r)/2);
- else if (tessPQ)
  - AdaptiveTessellate(p,(p+q)/2,r);
  - AdaptiveTessellate(q,r,(p+q)/2);
- Else if (not triTessEnough(p,q,r))

  AdaptiveTessellate((p+q+r)/3,p,q);

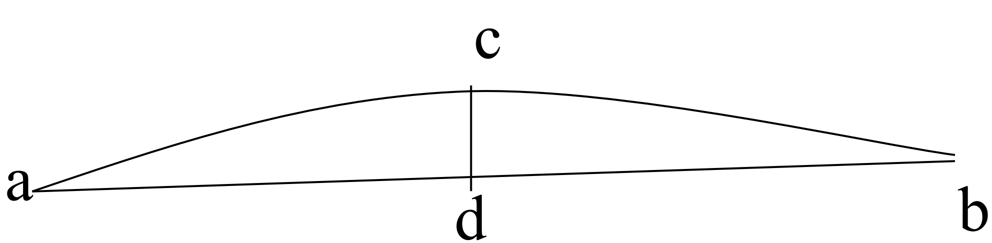
  AdaptiveTessellate((p+q+r)/3,q,r);

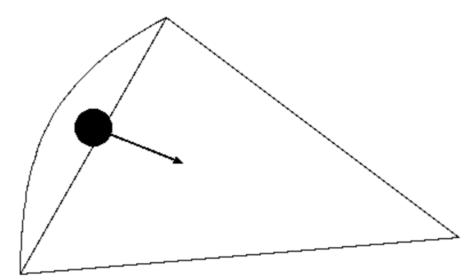
  AdaptiveTessellate((p+q+r)/3,r,p);

end;

curveTessEnough

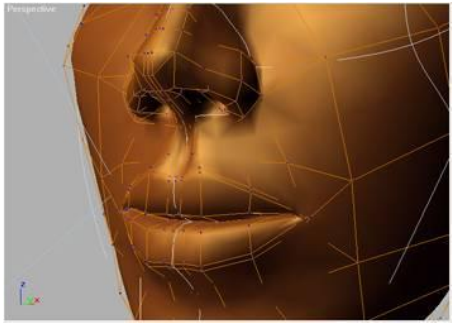
- Say you are to judge whether **ab** needs tessellation
- You can compute the midpoint c, and compute the curve's distance l from d, the midpoint of **ab**
- Check if l/||**a-b**|| is under a threshold
- Can do something similar for triTessEnough
  - Sample at the mass center and calculate its distance from the triangle





On-the-fly tessellation

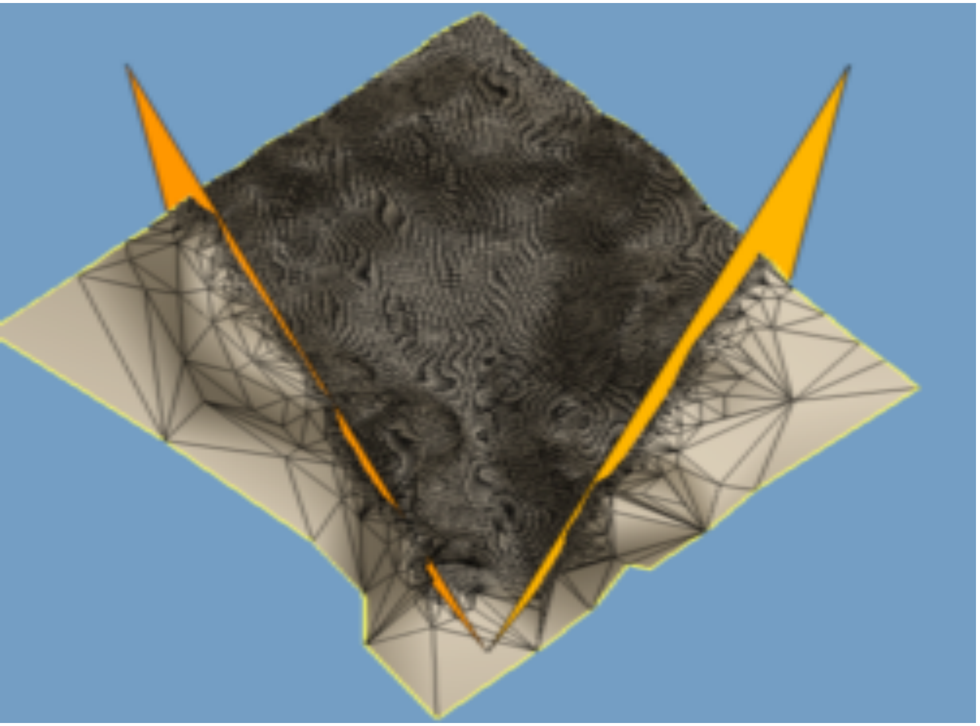
- In many cases, it is preferred to tessellate on-the-fly
- The size of the data can be kept small
- Tessellation is a highly parallel process
  - Can make use of the GPU
- The shape may deform in real-time

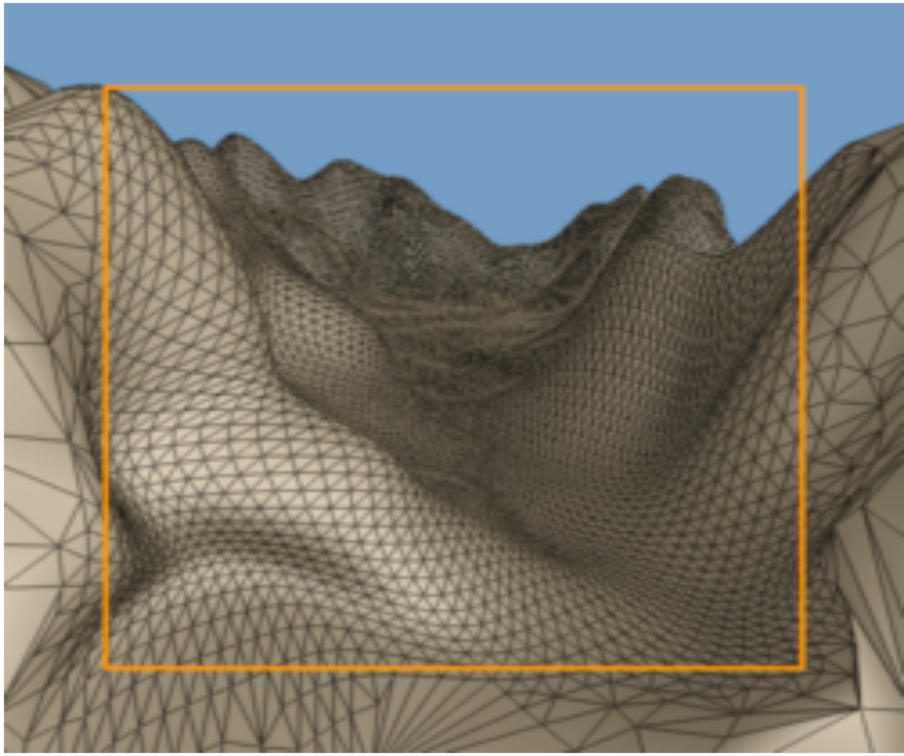


On-the-fly tessellation

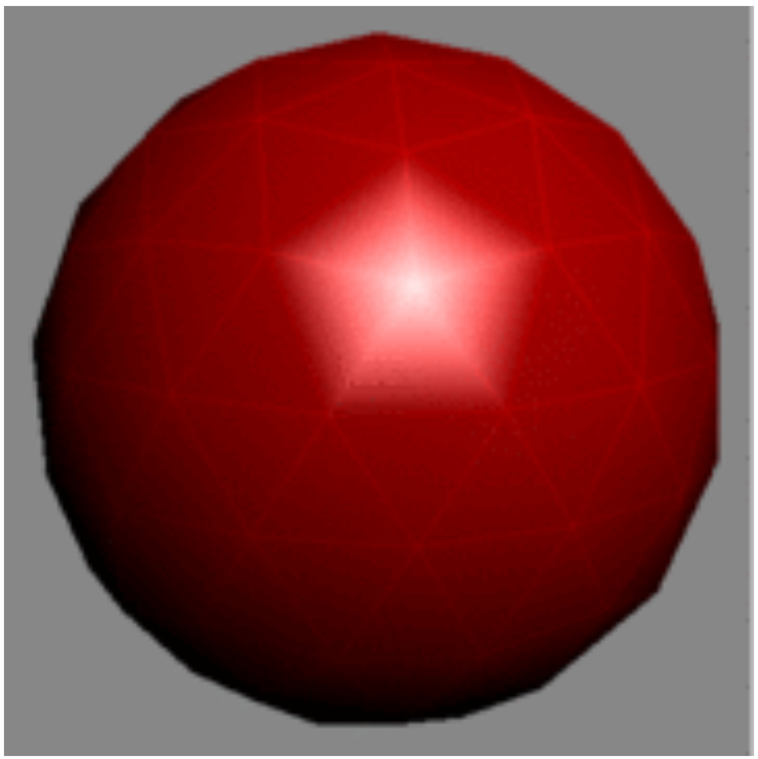
- So, say in a dynamic environment, what are the factors that we need to take into account when doing the tessellation?
  - in addition to curvature?

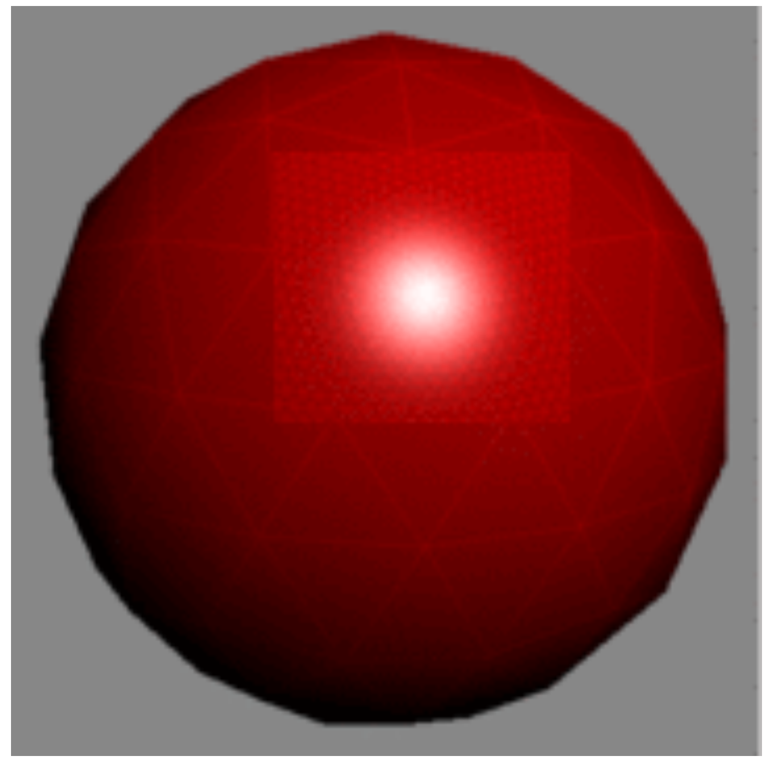
Other factors?



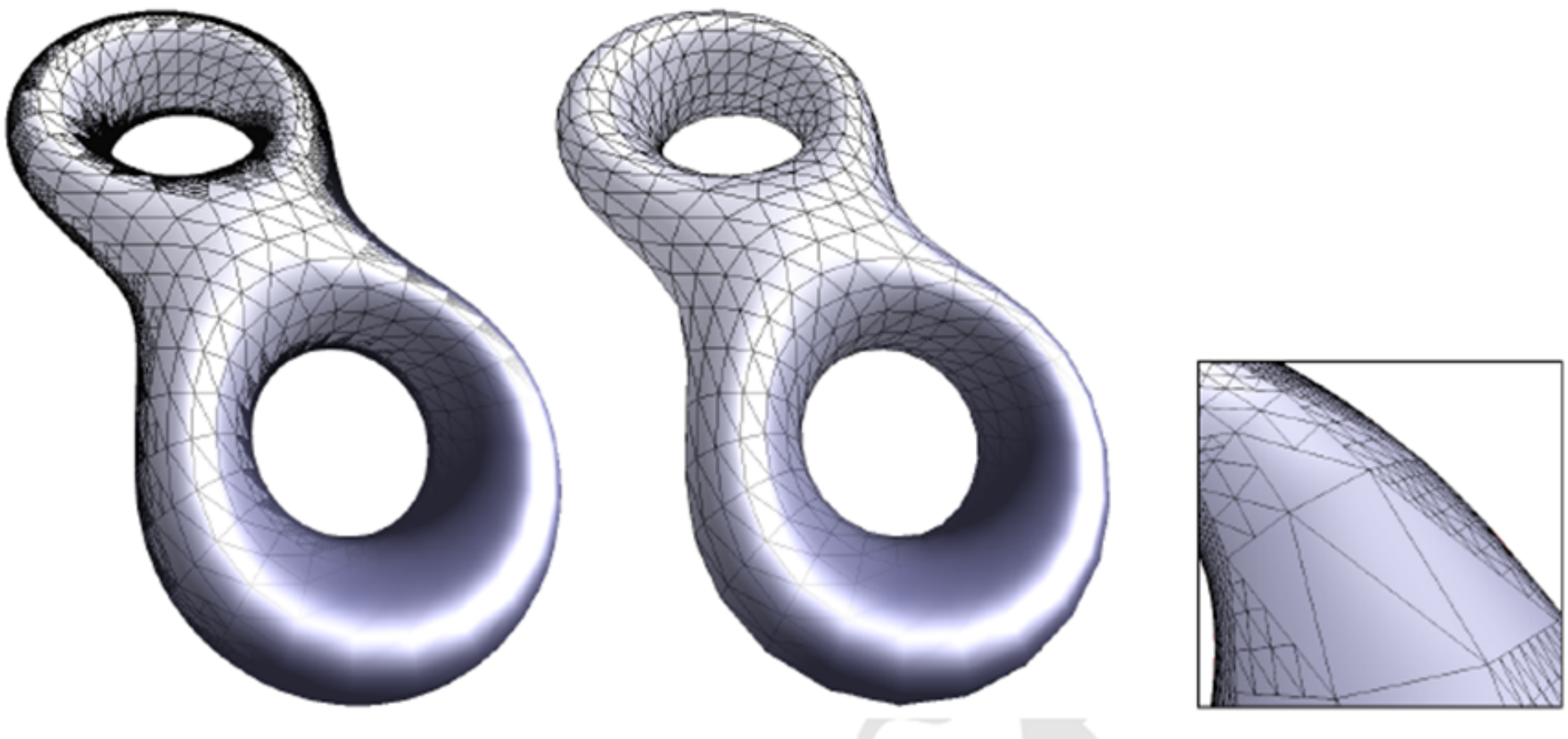


Other factors?





Other factors?



Other factors to evaluate

- Inside the view frustum
- Front facing
- Occupying a large area in screen space
- Close to the silhouette of the object
- Illuminated by a significant amount of specular lighting

Summary

- Hermite, Bezier, B-Spline curves
- Bicubic patches
- Tessellation
  - Triangulation of parametric surfaces
  - On-the-fly tessellation

Reading for this lecture

- Foley et al. Chapter 11, section 11.2 up to and including 11.2.3
- Introductory text Chapter 9, section 9.2 up to and including section 9.2.4
- Foley at al., Chapter 11, sections 11.2.3, 11.2.4, 11.2.9, 11.2.10, 11.3 and 11.5.
- Introductory text, Chapter 9, sections 9.2.4, 9.2.5, 9.2.7, 9.2.8 and 9.3.
- Real-time Rendering 2nd Edition Chapter 12.1-3