Curves

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How to create a virtual world?

To compose scenes

We need to define objects
- Characters
- Terrains
- Objects (trees, furniture, buildings etc)
Geometric representations

- Meshes
  - Triangle, quadrilateral, polygon

- Implicit surfaces
  - Blobs, metaballs

- Parametric surfaces / curves
  - Polynomials
  - Bezier curves, B-splines
Many applications require smooth surfaces

Can produce smooth surfaces with less parameters

- Easier to design
- Can efficiently preserve complex structures
From draftsmanship to CG

- Control
  - user specified control points
  - analogy: ducks

- Smoothness
  - smooth functions
  - usually low order polynomials
  - analogy: physical constraints, optimization
What is a curve?

A set of points that the pen traces over an interval of time

What is the dimensionality?

Implicit form: \( f(x,y) = x^2 + y^2 - 1 = 0 \)
- Find the points that satisfy the equation

Parametric form: \( (x,y) = f(t) = (cost, sint), \ t \in [0,2\pi) \)
- Easier to draw
What is a spline curve?

In this context

$f(t)$ is a

- parametric curve
- piecewise polynomial function that switches between different functions for different $t$ intervals

Example:

$$f(t) = \begin{cases} 
  t^3, & \text{if } 0 \leq t < 1 \\
  1 - (t - 1)^3, & \text{if } 1 \leq t < 2 \\
  0, & \text{otherwise}
\end{cases}$$
Defining spline curves

- Discontinuities at the integers \([t=k]\)
- Each spline piece is defined over \([k,k+1]\) (e.g. a cubic spline)
  \[
f(t) = at^3 + bt^2 + ct + d
\]
- Different coefficients for every interval
- Control of spline curves
  - Interpolate
  - Approximate
Today

- Spline segments
  - Linear
  - Quadratic
  - Hermite
  - Bezier
- Chaining splines

- Notation
  - vectors bold and lowercase $\mathbf{v}$
  - points as column vector $\mathbf{p} = (p_x \ p_y)$
  - matrices bold and uppercase $\mathbf{M}$
Spline segments

Linear Segment

A line segment connecting point $p_o$ to $p_1$

Such that $f(0) = p_0$ and $f(1) = p_1$

$$f_x(t) = (1 - t)x_o + tx_1$$

$$f_y(t) = (1 - t)y_o + ty_1$$

Vector formulation

$$f(t) = (1 - t)p_o + tp_1$$

Matrix formulation

$$f(t) = (t \ 1) \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

$$f(t) = (t \ 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$
Matrix form of spline

\[ f(t) = (t \quad 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \]

Blending functions \( b(t) \) specify how to blend the values of the control point vector

\[ f(t) = b_0(t)p_0 + b_1(t)p_1 \]

\( b_0(t) = 1 - t \)

\( b_1(t) = t \)
Beyond line segment

Quadratic

A quadratic \((f(t) = a_0 + a_1 t + a_2 t^2)\) passes through \(p_0, p_1, p_2\) s.t.

\[ p_0 = f(0) = a_0 + 0 \quad a_1 + 0^2 \quad a_2 \]
\[ p_1 = f(0.5) = a_0 + 0.5 a_1 + 0.5^2 a_2 \]
\[ p_2 = f(1) = a_0 + 1 \quad a_1 + 1^2 \quad a_2 \]

Points can be written in terms of constraint matrix \(C\)

\[
\begin{pmatrix}
  p_0 \\
p_1 \\
p_2
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 \\
  1 & 0.5 & 0.25 \\
  1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
  a_0 \\
a_1 \\
a_2
\end{pmatrix} \Rightarrow \begin{pmatrix}
  a_0 \\
a_1 \\
a_2
\end{pmatrix} = C^{-1} \begin{pmatrix}
p_0 \\
p_1 \\
p_2
\end{pmatrix}
\]

\(f(t)\) can be written in terms of basis matrix \(B = C^{-1}\) and points \(p\)

\[
f(t) = tBp = tC^{-1}p = \begin{pmatrix}
t^2 & t & 1
\end{pmatrix} \begin{pmatrix}
  2 & -4 & 2 \\
  -3 & 4 & -1 \\
  1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
p_0 \\
p_1 \\
p_2
\end{pmatrix}
\]
Matrix form of spline

Blending functions

\[ f(t) = (t^2 \ t \ 1) \begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} \]

Blending functions \( b(t) \) specify how to blend the values of the control point vector

\[
\begin{align*}
    f(t) &= b_0(t)p_0 + b_1(t)p_1 + b_2(t)p_2 \\
    b_0(t) &= 2t^2 - 3t + 1 \\
    b_1(t) &= -4t^2 - 4t \\
    b_2(t) &= 2t^2 - 1
\end{align*}
\]
Hermite spline

- Piecewise cubic \( f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \)
- Additional constraint on tangents (derivatives)
  - \( f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \)
  - \( f'(t) = a_1 + 2a_2 t + 3a_3 t^2 \)
  - \( p_0 = f(0) = a_0 \)
  - \( p_1 = f(1) = a_0 + a_1 + a_2 + a_3 \)
  - \( v_1 = f'(0) = a_1 \)
  - \( v_2 = f'(1) = a_1 + 2a_2 + 3a_3 \)
- Simpler matrix form

\[
f(t) = (t^3 \ t^2 \ t \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ v_1 \\ v_2 \end{pmatrix}
\]
Specify tangents as points:

- \( p_0 = q_0, p_1 = q_3, v_0 = 3(q_1 - q_0), v_1 = 3(q_3 - q_2) \)

\[
\begin{bmatrix}
  p_0 \\
p_1 \\
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
-3 & 3 & 1 & 0 \\
  0 & 0 & -3 & 3
\end{bmatrix}
\begin{bmatrix}
  q_0 \\
  q_1 \\
  q_2 \\
  q_3
\end{bmatrix}
\]

- Update Hermite eq. (from previous slide)

\[
f(t) = (t^3 \ t^2 \ t \ 1) \begin{bmatrix}
  2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
 0 & 0 & 1 & 0 \\
 2 & 0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
-3 & 3 & 1 & 0 \\
  0 & 0 & -3 & 3
\end{bmatrix}
\begin{bmatrix}
  q_0 \\
  q_1 \\
  q_2 \\
  q_3
\end{bmatrix}
\]
Bézier matrix

\[ f(t) = (t^3 \ t^2 \ t \ 1) \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 2 & 0 & -6 & 6 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \]

• \( f(t) = \sum_{n=0}^{d} b_{n,3} q_n \)

• Blending functions \( b(t) \) has a special name in this case:

• Bernstein polynomials

\[ b_{n,k} = \binom{n}{k} t^k (1 - t)^{n-k} \]

and that defines Bézier curves for any degree
Bézier blending functions

- The functions sum to 1 at any point along the curve.
- Endpoints have full weight.
Another view to Bézier segments

de Casteljau algorithm

Blend each linear spline with $\alpha$ and $\beta = 1 - \alpha$

$$\alpha p_0 + \beta p_1$$

$$\alpha^2 p_0 + 2\alpha\beta p_1 + \beta^2 p_1$$
Review

http://www.inf.ed.ac.uk/teaching/courses/cg/d3/hermite.html

http://www.inf.ed.ac.uk/teaching/courses/cg/d3/bezier.html

http://www.inf.ed.ac.uk/teaching/courses/cg/d3/Casteljau.html