Computational Foundations of Cognitive Science 1 (2009–2010)

School of Informatics, University of Edinburgh
Lecturers: Frank Keller, Miles Osborne

Solutions for Tutorial 7: Random Variables; Distributions and Densities

Week 8 (01–05 March 2010)

1. **Probability Distributions and Probability Densities**

(a) For each of the following, determine whether the given function can serve as the probability distribution for a random variable with the given range.

i. $f(x) = \frac{x-2}{5}$ for $x = 1,2,3,4,5$

ii. $f(x) = \frac{x^2}{30}$ for $x = 0,1,2,3,4$

iii. $f(x) = \frac{1}{5}$ for $x = 0,1,2,3,4,5$

Solution:

i. No, because $f(1) = \frac{1-2}{5} = -\frac{1}{5} < 0$

ii. Yes, because $\sum_{x=0}^{4} f(x) = \frac{0^2}{30} + \frac{1^2}{30} + \frac{2^2}{30} + \frac{3^2}{30} + \frac{4^2}{30} = 0 + \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30} = \frac{30}{30} = 1$

iii. No, because $\sum_{x=0}^{5} f(x) = 6 \cdot \frac{1}{5} = \frac{6}{5} > 1$

(b) Let X be the random variable indicating the difference between the number of heads and the number of tails obtained when tossing a balanced coin four times. Find the probability distribution of X.

Solution:

X	-4	-2	0	2	4
$P(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(c) Find the cumulative distribution $F(x)$ for the random variable with the probability distribution:

$$f(x) = \frac{x}{15} \text{ for } x = 1,2,3,4,5$$

Solution:

$$F(x) = \sum_{t \leq x} f(t) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{15} & \text{for } 1 \leq x < 2 \\ \frac{3}{15} & \text{for } 2 \leq x < 3 \\ \frac{6}{15} & \text{for } 3 \leq x < 4 \\ \frac{10}{15} & \text{for } 4 \leq x < 5 \\ 1 & \text{for } x \geq 5 \end{cases}$$

(d) The probability that a basketball player will score a goal at any given attempt is $\frac{2}{3}$. Assume that he attempts to score three times and compute:

i. the probability distribution of X, the total number of goals

ii. the probability of scoring at most two goals

Solution:

i.

X	0	1	2	3
$P(X)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

ii. $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{19}{27}$

(e) The probability density function of the random variable X is given by:

$$f(x) = \begin{cases} \frac{1}{8}(x+1) & \text{for } 2 \leq x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find $f(2 < X < 3)$.
ii. Find the cumulative distribution $F(x)$.

Solution:

i. Compute the integral of $f(x)$ for $2 < X < 3$:

$$\begin{aligned} P(2 \leq X \leq 3) &= \int_2^3 f(x)dx = \int_2^3 \frac{1}{8}(x+1)dx = \int_2^3 \frac{1}{8}x + \frac{1}{8}dx = \frac{1}{16}x^2 + \frac{1}{8}x \bigg|_2^3 \\ &= (\frac{3^2}{16} + \frac{3}{8}) - (\frac{2^2}{16} + \frac{2}{8}) = \frac{7}{16} \end{aligned}$$

ii. Compute the integral of $f(x)$ for $-\infty < X < x$:

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t)dt = \int_2^x \frac{1}{8}x + \frac{1}{8}dx = \frac{1}{16}x^2 + \frac{1}{8}x \bigg|_2^x \\ &= (\frac{1}{16}x^2 + \frac{1}{8}x) - (\frac{2^2}{16} + \frac{2}{8}) = \frac{1}{16}x^2 + \frac{1}{8}x - \frac{1}{2} \end{aligned}$$

Therefore the cumulative distribution function is:

$$F(x) = \begin{cases} 0 & \text{for } x \leq 2 \\ \frac{1}{16}x^2 + \frac{1}{8}x - \frac{1}{2} & \text{for } 2 < x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$

2. **Special Distributions**

(a) A scientist claims that 1 in 10 car accidents are due to driver fatigue. Using the formula for the binomial distribution, compute the probability that at most 3 of 5 accidents that happen on a given day are due to driver fatigue.

Solution:

$$b(0;5,0.1) + b(1;5,0.1) + b(2;5,0.1) + b(3;5,0.1) =$$

$$\binom{5}{0}0.1^0 0.9^5 + \binom{5}{1}0.1^1 0.9^4 + \binom{5}{2}0.1^2 0.9^3 + \binom{5}{3}0.1^3 0.9^2 = 0.9961$$

(b) A psychologist conducts a reaction time experiment, and then normalizes the resulting data so that the reaction times are distributed according to the standard normal distribution. (Note this means that there can be negative reaction times.)

What is the probability of obtaining a reaction time between 0 and 1 seconds?

Solution:

$$P(0 < X < 1) = \int_0^1 n(x;0,1)dx = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0.3413$$

We can work this out by integrating, or because we know that for the normal distribution, $P(|x-\mu| < \sigma) = 0.6826$, hence for the standard normal distribution $P(x < 1) = 0.6826/2 = 0.3413$.