1. Computing Eigenvectors

(a) Confirm that $x$ is a an eigenvector of $A$, and find the corresponding eigenvalue.

$$x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

**Solution:** We have

$$Ax = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5x,$$

thus $x$ is a an eigenvector of $A$ corresponding to the eigenvalue $\lambda = 5$.

(b) Find the characteristic equations of the following matrices, and then find their eigenvalues and eigenvectors.

$$B = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}.$$  

**Solution:** The characteristic equation of $B$ is $\det(\lambda I - B) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 1) = 0$. Thus $\lambda = 3$ and $\lambda = -1$ are the eigenvalues of $B$. To obtain the eigenvectors, we solve $(\lambda I - B)x = 0$, hence $\begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We substitute $\lambda = 3$ and get $\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which corresponds to the linear equations $0x + 0y = 0$ and $-8x + 4y = 0$, which have as solution $2x = y$. The first eigenvector is therefore $x = \begin{bmatrix} t \\ 2t \end{bmatrix}$. In the same way, we obtain the second eigenvector by solving $-4x + 0y = 0$ and $-8x + 0y = 0$, yielding $x = \begin{bmatrix} 0 \\ t \end{bmatrix}$.

The characteristic equation of $C$ is $\det(\lambda I - C) = \begin{vmatrix} \lambda - 10 & 9 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda - 10)(\lambda + 2) + 36 = (\lambda - 4)^2 = 0$. Thus $\lambda = 4$ is the only eigenvalue. The eigenvector is $x = \begin{bmatrix} t \\ 2 \end{bmatrix}$.

(c) Find the eigenvalues of the following matrices.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}.$$  

**Solution:** The eigenvalues of a of a triangular matrix are the elements on its diagonal. Hence the eigenvalues of $D$ are $\lambda = 3$, $\lambda = 7$, and $\lambda = 1$. The eigenvalues of $E$ are $\lambda = -\frac{1}{2}$, $\lambda = 1$, and $\lambda = \frac{1}{7}$.

(d) Using the eigenvalues you computed in questions (b) and (c), compute the determinants and traces of matrices $B$ to $E$. 


Solution: The determinant is the product of the eigenvalues of a matrix, the trace is the sum of the eigenvalues. Hence $\det(B) = 3(-1) = -3$, $\det(B) = 3 + (-1) = 2$, $\det(C) = 4 \cdot 4 = 16$, $\text{tr}(C) = 4 + 4 = 8$, $\det(D) = 3 \cdot 7 \cdot 1 = 21$, $\text{tr}(D) = 3 + 7 + 1 = 11$, $\det(E) = (-\frac{1}{4})(-\frac{1}{4}) \cdot \frac{1}{4} = \frac{1}{18}$, $\text{tr}(E) = -\frac{1}{3} - \frac{1}{3} + 1 + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$. Note that eigenvalues that occur more than once need to be entered in the computation more than once (such as $\lambda = -\frac{1}{3}$ in $E$).

2. Properties of Eigenvectors

(a) Find some matrices whose characteristic polynomial is $p(\lambda) = \lambda(\lambda - 2)^2(\lambda + 1)$.

Solution: We can write the characteristic polynomial as $p(\lambda) = (\lambda - 0)(\lambda - 2)(\lambda - 2)(\lambda + 1)$. This shows the matrix has the eigenvectors $\lambda = 0$, $\lambda = 2$, $\lambda = 2$, $\lambda = -1$.

Any triangular matrix with these values on its diagonal (in any order) is a correct answer. An example is:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
$$

(b) Suppose that the characteristic polynomial of $A$ is $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. What is the size of $A$? Is $A$ invertible?

Solution: The size of a matrix is given by the degree of the characteristic polynomial. Here, $p(\lambda)$ is of degree 6, hence it describes a $6 \times 6$ matrix. $A$ is invertible, as $\det(A) = 1 \cdot 3^2 \cdot 4^3 = 576 \neq 0$.

(c) Suppose that $A$ is a $2 \times 2$ matrix with $\text{tr}(A) = \det(A) = 4$. What are the eigenvalues of $A$?

Solution: As we saw in the lecture, the characteristic equation of a $2 \times 2$ matrix is $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$. Here, $\lambda^2 - 4\lambda + 4 = 0$, which is equivalent to $(\lambda - 2)^2 = 0$ hence $\lambda = 2$ is the only eigenvalue of $A$.

(d) Find all $2 \times 2$ matrices for which $\text{tr}(A) = \det(A)$.

Solution: The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the condition $\text{tr}(A) = \det(A)$ iff $a + d = ad - bc$. If $d = 1$ then this equation is satisfied iff $bc = -1$, e.g., $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. If $d \neq 1$, then the equation is satisfied iff $a = \frac{d + bc}{d - 1}$, e.g., $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

3. Convolutions

(a) Compute the convolution $k \ast a$ for the following vectors. What is the function of the kernel $k$?

$k = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $a = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$

Solution: The resulting vector $b = k \ast a$ is of dimensionality $2 + 6 - 1 = 7$. Its elements are computed as $b_k = \sum_a k_a a_{x-a+1}$ (note that all summands with out of range indices are dropped from the sum).

- $b_1 = k_1 a_{1-1+1} + k_2 a_{1-2+1} = k_1 a_1 = -1$
- $b_2 = k_2 a_{2-1+1} + k_2 a_{2-2+1} = k_1 a_2 + k_2 a_1 = 0$
- $b_3 = k_3 a_{3-1+1} + k_2 a_{3-2+1} = k_1 a_3 + k_2 a_2 = 1$
- $b_4 = k_4 a_{4-1+1} + k_2 a_{4-2+1} = k_1 a_4 + k_2 a_3 = 0$
- $b_5 = k_5 a_{5-1+1} + k_2 a_{5-2+1} = k_1 a_5 + k_2 a_4 = -1$
\[ b_6 = k_6 a_{6-1} + k_2 a_{6-2} = k_1 a_6 + k_2 a_5 = 0 \]
\[ b_7 = k_3 a_{7-1} + k_2 a_{7-2} = k_2 a_6 = 1 \]
So the resulting vector is \( b = [-1 \ 0 \ 1 \ 0 \ -1 \ 0 \ 1] \). The kernel \( \mathbf{k} \) approximates the derivative: it is \(-1\) if the elements of the input vector decrease in value (negative slope, e.g., change from 1 to 0), and 1 when the input values increase (positive slope, e.g., change from 0 to 1), and 0 when the input values remain unchanged (zero slope).

(b) Compute the convolution \( f \ast g \) for the following functions.

\[
g(x) = \begin{cases} 
3 & \text{if } 0 \leq x \leq 4 \\
0 & \text{otherwise}
\end{cases} \quad f(x) = \begin{cases} 
-\frac{1}{2} & \text{if } -1 \leq x \leq 0 \\
\frac{1}{2} & \text{if } 0 < x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Solution:** Integrating \( g(x) \) yields \( G(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
3x & \text{if } 0 \leq x \leq 4 \\
12 & \text{if } x > 4
\end{cases} \).

The convolution \( f \ast g \) is defined as \( (f \ast g)(x) = \int_{-\infty}^{+\infty} f(u) g(x-u) du \). We can restrict the integration boundaries based on \( f(u) \), and then apply integration by substitution, which yields \( (f \ast g)(x) = -\frac{1}{2} \int_{-1}^{0} g(x-u) du + \frac{1}{2} \int_{0}^{1} g(x-u) du = \frac{1}{2} \int_{0}^{1} g(u) du - \frac{1}{2} \int_{1}^{x} g(u) du = \frac{1}{2} (G(x) - G(x+1)) - \frac{1}{2} (G(x-1) - G(x)) = G(x) - \frac{1}{2} G(x+1) - \frac{1}{2} G(x-1) \). The resulting function is therefore \( (f \ast g)(x) = \begin{cases} 
0 & \text{if } x \leq -1 \\
3x - \frac{3}{2} (x+1) & \text{if } -1 < x \leq 0 \\
3x - \frac{3}{2} (x+1) & \text{if } 0 < x \leq 1 \\
12 - \frac{3}{2} \cdot 12 - \frac{3}{2} (x-1) & \text{if } 1 < x \leq 3 \\
12 - \frac{3}{2} \cdot 12 - \frac{3}{2} (x-1) & \text{if } 3 < x \leq 4 \\
12 - \frac{3}{2} \cdot 12 - \frac{3}{2} (x-1) & \text{if } 4 < x \leq 5 \\
0 & \text{if } x > 5
\end{cases} \).

(c) In image processing, what is the function of the following kernels?

\[
K_1 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \end{bmatrix}.
\]

**Solution:** \( K_1 \) is a horizontal edge detector (the transpose of the vertical edge detector discussed in the lecture). It is basically a more sophisticated version of the derivative kernel in question (a). \( K_2 \) blurs the image by averaging a pixel with the four pixels above and below and to its left and right. \( K_3 \) does exactly the opposite: it sharpens the image by changing the value of each pixel so that it is more distinct from the values of the four pixels below, above, left, and right of it.