Computational Foundations of Cognitive Science 1 (-2009–2010)

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Solutions for Tutorial 5: Eigenvectors and Convolutions

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1. **Computing Eigenvectors**

   (a) Confirm that $\mathbf{x}$ is a an eigenvector of A, and find the corresponding eigenvalue.
   $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

   Solution: We have $A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\mathbf{x}$, thus $\mathbf{x}$ is a an eigenvector of A corresponding to the eigenvalue $\lambda = 5$.

   (b) Find the characteristic equations of the following matrices, and then find their eigenvalues and eigenvectors.
   $B = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, C = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$.

   Solution: The characteristic equation of B is $\det(\lambda I - B) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 1) = 0$. Thus $\lambda = 3$ and $\lambda = -1$ are the eigenvalues of B. To obtain the eigenvectors, we solve $(\lambda I - B)\mathbf{x} = \mathbf{0}$, hence $\begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We substitute $\lambda = 3$ and get: $\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which corresponds to the linear equations $0x + 0y = 0$ and $-8x + 4y = 0$, which have as solution $2x = y$. The first eigenvector is therefore $\mathbf{x} = \begin{bmatrix} t \\ 2t \end{bmatrix}$. In the same way, we obtain the second eigenvector by solving $-4x + 0y = 0$ and $-8x + 0y = 0$, yielding $\mathbf{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$.

   The characteristic equation of C is $\det(\lambda I - C) = \begin{vmatrix} \lambda - 10 & 9 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda - 10)(\lambda + 2) + 36 = (\lambda - 4)^2 = 0$. Thus $\lambda = 4$ is the only eigenvalue. The eigenvector is $\mathbf{x} = \begin{bmatrix} \frac{3}{2}t \\ t \end{bmatrix}$.

   (c) Find the eigenvalues of the following matrices.
   $D = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}, E = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$.

   Solution: The eigenvalues of a of a triangular matrix are the elements on its diagonal. Hence the eigenvalues of D are $\lambda = 3$, $\lambda = 7$, and $\lambda = 1$. The eigenvalues of E are $\lambda = -\frac{1}{3}$, $\lambda = 1$, and $\lambda = \frac{1}{2}$.

   (d) Using the eigenvalues you computed in questions (b) and (c), compute the determinants and traces of matrices B to E.

Solution: The determinant is the product of the eigenvalues of a matrix, the trace is the sum of the eigenvalues. Hence $\det(B) = 3(-1) = -3$, $\mathrm{tr}(B) = 3 + (-1) = 2$, $\det(C) = 4 \cdot 4 = 16$, $\mathrm{tr}(C) = 4 + 4 = 8$, $\det(D) = 3 \cdot 7 \cdot 1 = 21$, $\mathrm{tr}(D) = 3 + 7 + 1 = 11$, $\det(E) = (-\frac{1}{3})(-\frac{1}{3})1 \cdot \frac{1}{2} = \frac{1}{18}$, $\mathrm{tr}(E) = -\frac{1}{3} - \frac{1}{3} + 1 + \frac{1}{2} = \frac{4}{6}$. Note that eigenvalues that occur more than once need to be entered in the computation more than once (such as $\lambda = -\frac{1}{3}$ in E).

2. **Properties of Eigenvectors**

   (a) Find some matrices whose characteristic polynomial is $p(\lambda) = \lambda(\lambda - 2)^2(\lambda + 1)$.

   Solution: We can write the characteristic polynomial as $p(\lambda) = (\lambda - 0)(\lambda - 2)(\lambda - 2)(\lambda + 1)$. This shows that the matrix has the eigenvectors $\lambda = 0$, $\lambda = 2$, $\lambda = 2$, $\lambda = -1$. Any triangular matrix with these values on its diagonal (in any order) is a correct answer. An example is:

   $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   (b) Suppose that the characteristic polynomial of A is $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. What is the size of A? Is A invertible?

   Solution: The size of a matrix is given by the degree of the characteristic polynomial. Here, $p(\lambda)$ is of degree 6, hence it describes a 6×6 matrix. A is invertible, as $\det(A) = 1 \cdot 3^2 \cdot 4^3 = 576 \neq 0$.

   (c) Suppose that A is a 2×2 matrix with $\mathrm{tr}(A) = \det(A) = 4$. What are the eigenvalues of A?

   Solution: As we saw in the lecture, the characteristic equation of a 2×2 matrix is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$. Here, $\lambda^2 - 4\lambda + 4 = 0$, which is equivalent to $(\lambda - 2)^2 = 0$ Hence $\lambda = 2$ is the only eigenvalue of A.

   (d) Find all 2×2 matrices for which $\mathrm{tr}(A) = \det(A)$.

   Solution: The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the condition $\mathrm{tr}(A) = \det(A)$ iff $a + d = ad - bc$. If $d = 1$ then this equation is satisfied iff $bc = -1$, e.g., $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. If $d \neq 1$, then the equation is satisfied iff $a = \frac{d+bc}{d-1}$, e.g., $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

3. **Convolutions**

   (a) Compute the convolution $\mathbf{k} * \mathbf{a}$ for the following vectors. What is the function of the kernel $\mathbf{k}$?

   $\mathbf{k} = \begin{bmatrix} -1 & 1 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$

   Solution: The resulting vector $\mathbf{b} = \mathbf{k} * \mathbf{a}$ is of dimensionality $2 + 6 - 1 = 7$. Its elements are computed as $b_x = \sum_u k_u a_{x-u+1}$ (note that all summands with out of range indices are dropped from the sum).

   $b_1 = k_1 a_{1-1+1} + k_2 a_{1-2+1} = k_1 a_1 = -1$
   $b_2 = k_2 a_{2-1+1} + k_2 a_{2-2+1} = k_1 a_2 + k_2 a_1 = 0$
   $b_3 = k_3 a_{3-1+1} + k_2 a_{3-2+1} = k_1 a_3 + k_2 a_2 = 1$
   $b_4 = k_4 a_{4-1+1} + k_2 a_{4-2+1} = k_1 a_4 + k_2 a_3 = 0$
   $b_5 = k_5 a_{5-1+1} + k_2 a_{5-2+1} = k_1 a_5 + k_2 a_4 = -1$

$b_6 = k_6 a_{6-1+1} + k_2 a_{6-2+1} = k_1 a_6 + k_2 a_5 = 0$
$b_7 = k_7 a_{7-1+1} + k_2 a_{7-2+1} = k_2 a_6 = 1$

So the resulting vector is $\mathbf{b} = \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$. The kernel $\mathbf{k}$ approximates the derivative: it is -1 if the elements of the input vector decrease in value (negative slope, e.g., change from 1 to 0), and 1 when the input values increase (positive slope, e.g., change from 0 to 1), and 0 when the input values remain unchanged (zero slope).

(b) Compute the convolution $f * g$ for the following functions.

$$g(x) = \begin{cases} 3 & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}, \quad f(x) = \begin{cases} -\frac{1}{2} & \text{if } -1 \le x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution: Integrating $g(x)$ yields $G(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 3x & \text{if } 0 \le x \le 4 \\ 12 & \text{if } x > 4 \end{cases}$.

The convolution $f * g$ is defined as $(f * g)(x) = \int_{-\infty}^{+\infty} f(u) g(x-u) du$. We can restrict the integration boundaries based on $f(u)$, and then apply integration by substitution, which yields $(f * g)(x) = -\frac{1}{2} \int_{-1}^{0} g(x-u) du + \frac{1}{2} \int_{0}^{1} g(x-u) du = \frac{1}{2} \int_{x+1}^{x} g(u) du - \frac{1}{2} \int_{x}^{x-1} g(u) du = \frac{1}{2}(G(x) - G(x+1)) - \frac{1}{2}(G(x-1) - G(x)) = G(x) - \frac{1}{2}G(x+1) - \frac{1}{2}G(x-1)$. The resulting function is therefore $(f * g)(x) =$

$$\begin{cases} 0 & \text{if } x \le -1 \\ -\frac{3}{2}(x+1) & \text{if } -1 < x \le 0 \\ 3x - \frac{3}{2}(x+1) & \text{if } 0 < x \le 1 \\ 3x - \frac{3}{2}(x+1) - \frac{3}{2}(x-1) & \text{if } 1 < x \le 3 \\ 3x - \frac{1}{2} \cdot 12 - \frac{3}{2}(x-1) & \text{if } 3 < x \le 4 \\ 12 - \frac{1}{2} \cdot 12 - \frac{3}{2}(x-1) & \text{if } 4 < x \le 5 \\ 12 - \frac{1}{2} \cdot 12 - \frac{1}{2} \cdot 12 & \text{if } x > 5 \end{cases} = \begin{cases} 0 & \text{if } x \le -1 \\ -\frac{3}{2}x - \frac{3}{2} & \text{if } -1 < x \le 0 \\ \frac{3}{2}x - \frac{3}{2} & \text{if } 0 < x \le 1 \\ 0 & \text{if } 1 < x \le 3 \\ \frac{3}{2}x - \frac{9}{2} & \text{if } 3 < x \le 4 \\ -\frac{3}{2}x - \frac{9}{2} & \text{if } 4 < x \le 5 \\ 0 & \text{if } x > 5 \end{cases}.$$

(c) In image processing, what is the function of the following kernels?

$$K_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix}, K_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution: K_1 is a horizontal edge detector (the transpose of the vertical edge detector discussed in the lecture). It is basically a more sophisticated version of the derivative kernel in question (a). K_2 blurs the image by averaging a pixel with the four pixels above and below and to its left and right. K_3 does exactly the opposite: it sharpens the image by changing the value of each pixel so that it is more distinct from the values of the four pixels below, above, left, and right of it.