Computational Foundations of Cognitive Science 1 (2009–2010)

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Solutions for Tutorial 4: Inverses and Determinants

Week 5 (08–12 February 2010)

1. **Computing Inverses**

   Assume the following matrices:

   $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

   Compute the following matrices, where possible:

   (a) A^{-1}, B^{-1}

   Solution: $\det(A) = ad - bc = 3 \cdot 2 - 1 \cdot 5 = 1 \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

   $\det(B) = 8 + 12 = 20 \quad B^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$

   (b) $(A^T)^{-1}$

   Solution: $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

   (c) $(2A)^{-1}$

   Solution: $(2A)^{-1} = \frac{1}{2}A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$

   (d) $(AB)^{-1}$

   Solution: $(AB)^{-1} = B^{-1}A^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix}$

2. **Terms with Inverses**

   Assume that A, B, C, D are invertible matrices, such that the products given in (a) and (b) are defined.

   (a) Simplify the following term as much as possible: $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$
   Solution: $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1} = B^{-1}A^{-1}AC^{-1}C^{-1}DD^{-1} = B^{-1}$

   (b) Simplify the following term as much as possible: $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$
   Solution: $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1} = CA^{-1}AC^{-1}CA^{-1}AD^{-1} = CD^{-1}$

   (c) Find all values of c for which $A = \begin{bmatrix} -c & -1 \\ 1 & c \end{bmatrix}$ is invertible.

   Solution: A is invertible iff $\det(A) = -c^2 + 1 \neq 0$, i.e., iff $c \neq \pm 1$.

3. **Matrices with Special Forms**

   Assume the following matrices:

   $C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix}, G = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

   Compute the following matrices, where possible:

(a) C^{-1}, D^{-1}, G^{-1}

Solution: C is not invertible, as it has a row of zeros. D is a diagonal matrix, hence it can be inverted by taking the reciprocal of the elements on the diagonal: $D^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$. The inverse of a symmetric matrix is symmetric and is computed in the standard way using the determinant: $G^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) CD

Solution: $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 4(-1) & 0 & 0 \\ 0 & 0\cdot 2 & 0 \\ 0 & 0 & 5\cdot\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{5}{3} \end{bmatrix}$

(c) EF

Solution: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 & 3\cdot 1 \\ (-1)(-4) & (-1)1 \\ 2\cdot 2 & 2\cdot 5 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$

(d) Which of the following is symmetric: C, E, F^T, F^TF, G^{-1}.

Solution: C and E are symmetric by definition. F^T is not symmetric as it is not square. F^TF is symmetric as A^TA and AA^T are symmetric for any A. G^{-1} is symmetric as the inverse of a symmetric matrix is symmetric.

4. **Determinants**

Assume the matrices A to F in Questions 1 and 3, as well as:

$H = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$

(a) Compute $\det(A)$, $\det(A^{-1})$, $\det(A^T)$.

Solution: $\det(A) = ad - bc = 3\cdot 2 - 1\cdot 5 = 1 \quad \det(A^{-1}) = \frac{1}{\det(A)} = 1 \quad \det(A^T) = \det(A) = 1$

(b) Compute $\det(C)$, $\det(D)$, $\det(G)$.

Solution: For diagonal and triangular matrices, the determinant is the product of the entries on the main diagonal. Hence $\det(C) = 4\cdot 0\cdot 5 = 0 \quad \det(D) = (-1)2\cdot\frac{1}{3} = -\frac{2}{3}$. The determinants of symmetric matrices have no special properties, hence $\det(G) = 2\cdot 3 - (-1)(-1) = 5$.

(c) Find all values of λ for which $\det(H) = 0$.

Solution: $\det(H) = (\lambda-2)(\lambda+4)+5 = \lambda^2+2\lambda-3 = (\lambda-1)(\lambda+3)$. Thus $\det(H) = 0$ iff $\lambda = 1$ or $\lambda = -3$.