Computational Foundations of Cognitive Science 1 (2009–2010)

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Solutions for Tutorial 3: Matrix Operations

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1. **Matrix Addition and Subtraction**

   Assume the following matrices:

   $$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 4 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix}, E = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

   Compute the following matrices, where possible:

   (a) $A + 2B$

   Solution: $\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -7 & 4 \\ 9 & 1 \end{bmatrix}$

   (b) $A - B^T$

   Solution: not defined, as A is 3×2 and B^T is 2×3.

   (c) $4D - 3C^T$

   Solution: $4 \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -12 & 15 \end{bmatrix}$

   (d) $D - D^T$

   Solution:
   $\begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

2. **Multiplying a Matrix with a Vector**

   Assume the following: $A = \begin{bmatrix} 1 & 5 & 2 \\ -4 & 9 & 1 \\ 2 & 0 & 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Compute $A\mathbf{x}$.

   Solution: $2 \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -14 \\ 13 \end{bmatrix}$

3. **Matrix Multiplication**

   Assume the same matrices as in Question 1. Compute the following matrices, where possible:

   (a) CD

   Solution: $\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1(C)\mathbf{c}_1(D) & \mathbf{r}_1(C)\mathbf{c}_2(D) \\ \mathbf{r}_2(C)\mathbf{c}_1(D) & \mathbf{r}_2(C)\mathbf{c}_2(D) \end{bmatrix} =$
   $\begin{bmatrix} 1 \cdot 1 + 0(-3) & 1 \cdot 1 + 0 \cdot 3 \\ 3 \cdot 1 + (-1)(-3) & 3 \cdot 1 + (-1)3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$

(b) AE

Solution: $\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1(A)\mathbf{c}_1(E) & \mathbf{r}_1(A)\mathbf{c}_2(E) & \mathbf{r}_1(A)\mathbf{c}_3(E) \\ \mathbf{r}_2(A)\mathbf{c}_1(E) & \mathbf{r}_2(A)\mathbf{c}_2(E) & \mathbf{r}_2(A)\mathbf{c}_3(E) \\ \mathbf{r}_3(A)\mathbf{c}_1(E) & \mathbf{r}_3(A)\mathbf{c}_2(E) & \mathbf{r}_3(A)\mathbf{c}_3(E) \end{bmatrix} = \begin{bmatrix} 3 & 12 & 6 \\ 5 & -2 & 8 \\ 4 & 5 & 7 \end{bmatrix}$

(c) BB^T

Solution: $\begin{bmatrix} 5 & -5 & 8 \\ -5 & 10 & -12 \\ 8 & -12 & 16 \end{bmatrix}$

(d) DA

Solution: not defined, as D is 2×2 and A is 3×2.

4. **Inner and Outer Product**

Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

(a) Find the matrix inner product of $\mathbf{u}$ with $\mathbf{v}$.

Solution: $\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = (-2)4 + 3 \cdot 5 = 7$

(b) Find the matrix outer product of $\mathbf{u}$ with $\mathbf{v}$.

Solution: $\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} (-2)4 & (-2)5 \\ 3 \cdot 4 & 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} -8 & -10 \\ 12 & 15 \end{bmatrix}$

(c) Find the dot product of $\mathbf{u}$ with $\mathbf{v}$.

Solution: $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$

5. **Application**

The following tables shows a record of unit sales for a clothing store. Let M denote the 4×3 matrix of sales.

	Small	Medium	Large
Shirts	45	60	75
Jeans	30	30	40
Suits	12	65	45
Raincoats	15	40	25

(a) Find the column vector $\mathbf{x}$ for which $M\mathbf{x}$ provides a list of the number of shirts, jeans, suits, and raincoats sold.

Solution: Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then the components of $M\mathbf{x} = \begin{bmatrix} 45 & 60 & 75 \\ 30 & 30 & 40 \\ 12 & 65 & 45 \\ 15 & 40 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 180 \\ 100 \\ 122 \\ 80 \end{bmatrix}$ represent the total number of shirts, jeans, suits, and raincoats sold.

(b) Find the row vector $\mathbf{y}$ for which $\mathbf{y}M$ provides a list of small, medium, and large items sold.

Solution: Let $\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. Then the components of $\mathbf{y}M = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 45 & 60 & 75 \\ 30 & 30 & 40 \\ 12 & 65 & 45 \\ 15 & 40 & 25 \end{bmatrix} = \begin{bmatrix} 102 & 195 & 125 \end{bmatrix}$ represent the total number of small, medium, and large items sold.

(c) What does $\mathbf{y}M\mathbf{x}$ represent?

Solution: The product $\mathbf{y}M\mathbf{x} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}45 & 60 & 75\\ 30 & 30 & 40\\ 12 & 65 & 45\\ 15 & 40 & 25\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} =$ $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}180\\100\\122\\90\end{bmatrix} = 492$ represent the total number of items sold.

6. **Image Processing**

Assume that you have two matrices A and B representing greyscale images. A represents a picture of a lake and B a picture of a ship. If you compute $A+B$ and $A+B^T$, what do they represent? What do AB and BB^T represent?

Solution: $A+B$ represents the addition of the two pictures, i.e., a picture of a lake and a ship. $A+B^T$ is the same picture, but with the ship rotated by 90 degrees anticlockwise. AB and BB^T do not have a useful interpretation for these two images. Most likely the resulting images will look very bright (or even completely white), as matrix multiplication means that a lot of values will be larger than 255 (i.e., representing white).