Computational Foundations of Cognitive Science
Lecture 13: Determinants and Eigenvectors

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Reading: Anton and Busby, Chs. 4.1, 4.4
Determinants determine if a matrix is invertible (Lecture 12). They are also important for eigenvectors. Recall:

**Definition: Determinant of a 2 × 2 Matrix**

The determinant of a 2 × 2 matrix \( A \) is given by:

\[
\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

We can extend this to 3 × 3 matrices:

**Definition: Determinant of a 3 × 3 Matrix**

The determinant of a 3 × 3 matrix \( A \) is given by:

\[
\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}
\]
The following diagrams make these formulae easier to remember: the determinant consists of the products obtained by following the arrows, with the appropriate sign:

Example

\[
\begin{vmatrix}
1 & 2 & 3 \\
-4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix} = 45 + 84 - 96 - 105 - 48 - (-72) = -48
\]
Determinants
Eigenvalues and Eigenvectors

Geometric Interpretation of Determinants

**Theorem: Geometric Interpretation of Determinants**

If $A$ is a $2 \times 2$ matrix, then $\text{abs}(\det(A))$ represents the area of the parallelogram determined by the two column vectors of $A$ when they are positioned so their initial points coincide.

Note: $\text{abs}(x)$ is the absolute value of $x$.

**Example**

\[
A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix} = [u \ v]. \text{ The area of the parallelogram determined by } u \text{ and } v \text{ is:} \\
\text{abs}(\det(A)) = \text{abs}(\begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix}) = \text{abs}(-28) = 28.
\]
General Determinants

This method only works for $2 \times 2$ and $3 \times 3$ matrices. For larger matrices, we need a more general definition.

An *elementary product* from an $n \times n$ matrix $A$ is a product of $n$ entries from $A$, where no two come from the same row or column:

$$\pm a_{j_1}a_{j_2}\cdots a_{j_n}$$

Here, the column indices $\{j_1, j_2, \ldots, j_n\}$ form a permutation of the integers from 1 to $n$.

The sign of the elementary product is based on the minimum number of interchanges required to put the permutation in natural order. If it is even, then the sign is $+$, if it is odd, then $-$.
General Determinants

Definition: Determinant

The determinant of a square matrix $A$ is denoted by $\det(A)$ and is defined to be sum of all elementary products from $A$:

$$\det(A) = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

Computing determinants based on this definition is computationally intractable for large $n$, as the number of permutations is:

$$n! = n(n-1)(n-2) \cdots 1$$

This figure grows very quickly: $3! = 6$, $4! = 24$, $5! = 120$, $10! = 3,628,800$. More efficient algorithms are available, and are used in computational tools such as Matlab.
Theorem: Determinants for Matrices with Zero Rows or Columns

If $A$ is a square matrix with a row or a column of zeros, then $\det(A) = 0$.

Each elementary product has an entry from each row, therefore each product has a factor that is zero, hence the product is zero. So the sum of all elementary products is zero. Same for columns.

Theorem: Determinants for Triangular Matrices

If $A$ is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal.

The only elementary product that is not zero is $\pm a_{11} a_{22} \cdots a_{nn}$, which is the product of the entries on the main diagonal.
Determinants of Matrices with Special Form

Example

\[
\begin{vmatrix}
-2 & 5 & 7 \\
0 & 3 & 8 \\
0 & 0 & 5
\end{vmatrix} = (-2) \cdot 3 \cdot 5 = -30
\]

\[
1 \cdot 9(-1)(-2) = 18
\]

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
4 & 9 & 0 & 0 \\
-7 & 6 & -1 & 0 \\
3 & 8 & -5 & -2
\end{vmatrix} =
\]

A computationally more efficient way of computing an \( n \times n \) determinant is *cofactor expansion*, which expresses this determinant in terms of \((n - 1) \times (n - 1)\) determinants.

Cofactor expansion can be applied recursively until we have a set of \( 2 \times 2 \) determinants to compute. See Anton and Busby, Ch. 4.1.
**Definition: Eigenvalues and Eigenvectors**

If $A$ is an $n \times n$ matrix, then a scalar $\lambda$ is called an eigenvalue of $A$ if there is a nonzero vector $\mathbf{x}$ (called eigenvector) such that $A\mathbf{x} = \lambda \mathbf{x}$.

We compute the eigenvalues of $A$ by solving the *characteristic equation* of $A$: $\det(\lambda I - A) = 0$

**Example**

To find the eigenvalues of $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, first compute:

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix}.$$  

Then the characteristic equation is:

$$\begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = 0,$$

which yields: $\lambda^2 - 3\lambda - 10 = 0$, which has the solutions $\lambda = -2$ and $\lambda = 5$. 
Once we have the eigenvalues, we can compute the eigenvectors by solving the equation $Ax = \lambda x$, equivalent to: $(\lambda I - A)x = 0$.

**Example**

For the previous example, the equation $(\lambda I - A)x = 0$ is:

$$
\begin{bmatrix}
\lambda - 1 & -3 \\
-4 & \lambda - 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

We substitute $\lambda = -2$ and obtain:

$$
\begin{bmatrix}
-3 & -3 \\
-4 & -4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

Hence we need to solve the following system of linear equations:

$$
\begin{align*}
-3x + -3y &= 0 \\
-4x + -4y &= 0
\end{align*}
$$

which yields the solutions $x = -t$ and $y = t$, hence the eigenvector is $\begin{bmatrix} -t \\ t \end{bmatrix}$. The eigenvector for $\lambda = 5$ is $\begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix}$ (details omitted).
By definition, the eigenvalues and eigenvectors of $A$ make the following equation true: $A\mathbf{x} = \lambda \mathbf{x}$.

We can verify this by substituting $\mathbf{x}$ into this equation.

**Example**

First eigenvector of $A$:

$A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \end{bmatrix} = -2 \begin{bmatrix} -t \\ t \end{bmatrix} = -2\mathbf{x}$

Second eigenvector of $A$:

$A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{15}{4}t \\ \frac{5}{4}t \end{bmatrix} = 5 \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = 5\mathbf{x}$
Suppose $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ is a matrix with the eigenvector $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and the corresponding eigenvalue $\lambda = 4$. Verify that $\lambda^2 = 16$ is an eigenvalue of $A^2 = AA$, for the same eigenvector.

Is this a general property? In other words, if $\lambda$ is an eigenvalue of $A$, then is $\lambda^2$ always an eigenvalue of $A^2$?
Let’s use Matlab’s `eigshow` to find the eigenvectors of \( A = \begin{bmatrix} 5 & 0 \\ 4 & 3/4 \end{bmatrix} \):

If \( \mathbf{x} \) is parallel to \( A\mathbf{x} \), then \( \mathbf{x} \) is an eigenvector. Here: \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \lambda = \frac{5}{4} \).
Eigenvalues of Triangular Matrices

If $A$ is a triangular matrix with the diagonal $a_{11}, a_{22}, \ldots, a_{nn}$, then $\lambda I - A$ is triangular with diagonal $\lambda - a_{11}, \lambda - a_{22}, \ldots, \lambda - a_{nn}$. The characteristic polynomial of $A$ is:

$$\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$$

which implies that the eigenvalues of $A$ are:

$$\lambda_1 = a_{11}, \lambda_2 = a_{22}, \ldots, \lambda_n = a_{nn}$$

Theorem: Eigenvalues of Triangular Matrices

If $A$ is a triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $A$ are the entries on the main diagonal of $A$.

Example: on previous slide.
The characteristic equation of a $2 \times 2$ matrix is:

$$\det(\lambda I - \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 0$$

This can be written as:

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

because we can make the following simplification:

$$\det(\lambda I - \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc =$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$
Eigenvalues of $2 \times 2$ Matrices

From algebra we know that a quadratic equation $ax^2 + bx + c = 0$ has two solutions if $b^2 - 4ac > 0$, one solution if $b^2 - 4ac = 0$, and no solutions if $b^2 - 4ac < 0$.

We set $a = 1$, $b = -\text{tr}(A)$, and $c = \det(A)$ and obtain:

**Theorem: Eigenvalues of $2 \times 2$ Matrices**

If $A$ is a $2 \times 2$ matrix, then the characteristic equation of $A$ is $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$ and

- $A$ has two eigenvalues if $\text{tr}(A)^2 - 4\det(A) > 0$
- $A$ has one eigenvalue if $\text{tr}(A)^2 - 4\det(A) = 0$
- $A$ has no eigenvalues if $\text{tr}(A)^2 - 4\det(A) < 0$

Note: in the case of $\text{tr}(A)^2 - 4\det(A) < 0$, $A$ has imaginary eigenvalues. We will ignore this.
Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

We have $\text{tr}(A) = 7$ and $\det(A) = 12$, so the characteristic equation of $A$ is $\lambda^2 - 7\lambda + 12 = 0$. We solve this using the quadratic formula and get: $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{(-7)^2 - 4 \cdot 12}}{2}$, so $\lambda = 4$ and $\lambda = 3$.

We have $\text{tr}(B) = 2$ and $\det(B) = 1$ and a characteristic equation $\lambda^2 - 2\lambda + 1 = 0$, which yields $\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4}}{2} = 1$. 
There are formulae for computing determinants for $2 \times 2$ and $3 \times 3$ matrices;

- Elementary products can be used to compute general determinants (but: very inefficient);
- $\det(A) = 0$ if $A$ has a row or column of zeros;
- $\det(A) = a_{11} a_{22} \cdots a_{nn}$ if $A$ is triangular;
- If $A\mathbf{x} = \lambda \mathbf{x}$, then $\lambda$ is an eigenvalue and $\mathbf{x}$ an eigenvector;
- They can be computed by solving the characteristic equation of $A$, $\det(\lambda I - A) = 0$;
- The eigenvalues of a triangular matrix are the entries on its diagonal;
- The characteristic equation of a $2 \times 2$ matrix is $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$. 