Computational Foundations of Cognitive Science

Lecture 12: Inverses; Matrices with Special Forms

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Reading: Anton and Busby, Chs. 3.2, 3.6

Definition of the Inverse

Every real number a has a reciprocal $a^{-1} = 1/a$ such that $a \cdot a^{-1} = 1$. The analogue for a matrix A is its *inverse* A^{-1}.

Definition: Matrix Inverse

If A is a square matrix, and if there is a matrix B with the same size such that $AB = I$, then A is invertible (non-singular), and B is the inverse of A. If there is no such matrix B, then A is singular.

Note that if $AB = I$, then also $BA = I$, i.e., B is also invertible and an inverse of A.

Example

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Definition of the Inverse

In general, a square matrix with a row or column of zeros is singular, i.e., it has no inverse.

The row or column of zeros causes the result of matrix multiplication to also have a row or column of zeros, i.e., it can't be I.

Example

Assume $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{0} \end{bmatrix}$.

Then apply the definition of matrix multiplication to compute $BA = B\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} B\mathbf{c}_1 & B\mathbf{c}_2 & \mathbf{0} \end{bmatrix}$. This shows that $BA \neq I$.

Definition of the Inverse

Theorem: Uniqueness of the Inverse

If A is an invertible matrix, and if B and C are both inverses of A, then $B = C$, i.e., an invertible matrix has a unique inverse.

The uniqueness of the inverse implies $AA^{-1} = I$ and $A^{-1}A = I$.

Finding the inverse of a matrix is generally non-trivial. However, for 2×2 matrices, there is a simple formula. For this we need the notion of a *determinant* of a matrix.

Definition: Determinant of a Matrix

The determinant of a 2×2 matrix A is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Computation of the Inverse

Theorem: Computation of the Inverse

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $\det(A) \neq 0$, in which case:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Examples

$$A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

$\det(A) = 6 \cdot 2 - 1 \cdot 5 = 7 \quad \det(B) = (-1)(-6) - 2 \cdot 3 = 0$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

Properties of the Inverse

Theorem: Product of Inverses; Transpose

If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \quad B^{-1}A^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

In general, a product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is a *diagonal matrix:*

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all the d_i are non-zero in which case the inverse is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Diagonal Matrices

Matrix products involving diagonal matrices are easy to compute:

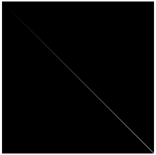
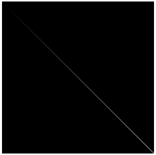
Examples

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

Example: Representing Images

Let us again represent matrices as greyscale images. Let A be a diagonal matrix with the values $1 \ldots 200$ on the diagonal:

$A =$  $A^T =$ 

Transposing a diagonal matrix doesn't change it.

Mid-lecture Problem

Indicate whether the following statements are true or false. Justify your answers.

1. If A is invertible, then so is $I - A$.
2. If A is a diagonal matrix, then so is AA^T.
3. If A and B are diagonal matrices, then $AB = BA$.

Triangular Matrices

A square matrix is *triangluar* if all entries to the left of the main diagonal are zero (upper triangular), or if all entries to the right of main diagonal are zero (lower triangular). More formally:

- $A = [a_{ij}]$ is upper triangular iff $a_{ij} = 0$ for all $i > j$;
- $A = [a_{ij}]$ is lower triangular iff $a_{ij} = 0$ for all $i < j$.

Matrix can be both upper and lower triangular (e.g., diagonal matrix).

Example

A 4×4 upper and lower triangular matrix has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{23} & a_{33} & 0 \\ a_{41} & a_{24} & a_{43} & a_{44} \end{bmatrix}$$

Triangular Matrices

Theorem: Properties of Triangular Matrices

- The transpose of a lower (upper) triangular matrix is upper (lower) triangular.
- A product of lower (upper) triangular matrices is lower (upper) triangular.
- A triangular matrix is invertible iff its diagonal entries are all non-zero.
- The inverse of a lower (upper) triangular matrix is lower (upper) triangular.

Triangular Matrices

Example

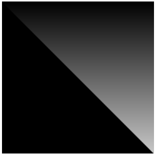
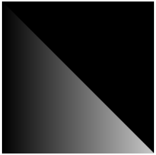
$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

A is invertible but B isn't. Also, A^{-1}, AB and BA must be upper triangular:

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

Example: Representing Images

Now let B be an upper triangular matrix with the values $1 \dots 200$ on the rows $1 \dots 200$:

$B =$  $B^T =$ 

Transposing an upper triangular matrix yields a lower triangular matrix.

Symmetric Matrices

A square matrix A is *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$. More formally:

- $A = [a_{ij}]$ symmetric iff $(A)_{ij} = (A)_{ji}$, or equivalently, $a_{ij} = a_{ji}$;
- $A = [a_{ij}]$ skew-symmetric iff $(A)_{ij} = -(A)_{ji}$, or equivalently, $a_{ij} = -a_{ji}$.

Examples

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & -6 \\ 5 & -6 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -4 & 5 \\ 4 & 0 & -9 \\ -5 & 9 & 0 \end{bmatrix}$$

Symmetric Matrices

Theorem: Properties of Symmetric Matrices

If A and B are symmetric matrices with the same size, and k is a scalar, then:

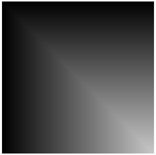
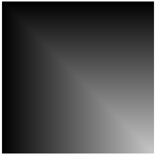
- A^T is symmetric
- $A+B$ and $A-B$ are symmetric
- kA is symmetric
- AB is symmetric iff A and B commute, i.e., iff $AB = BA$
- if A is symmetric and invertible, then A^{-1} is symmetric

Examples

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$$

Example: Representing Images

We get a symmetric matrix C by adding B and B^T:

$C =$  $C^T =$ 

Transposing a symmetric matrix doesn't change it.

Matrices of the form AA^T and A^TA

If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, which means that AA^T and A^TA are square matrices of size $m \times m$ and $n \times n$.

AA^T and A^TA are always symmetric, since:
$(AA^T)^T = (A^T)^TA^T = AA^T$ and $(A^TA)^T = A^T(A^T)^T = A^TA$.

Example

$$A^TA = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{bmatrix}$$

By definition of the inner product,
$\mathbf{a}_1^T\mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_2 \cdot \mathbf{a}_1 = \mathbf{a}_2^T\mathbf{a}_1$, hence the matrix is symmetric.

Summary

- The inverse of a matrix A is A^{-1} with $AA^{-1} = I$;
- the determinant of a 2×2 matrix is $\det(A) = ad - bc$;
- $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$;
- diagonal matrix: all entries off the main diagonal are zero;
- triangular matrix: all entries to the left or right of the main diagonal are zero;
- symmetric matrix: $A^T = A$;
- AA^T and A^TA are always symmetric.