# Hidden Markov Models and Gaussian Mixture Models

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Automatic Speech Recognition— ASR Lectures 4&5
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# Overview

## HMMs and GMMs

- Key models and algorithms for HMM acoustic models
- Gaussians
- GMMs: Gaussian mixture models
- HMMs: Hidden Markov models
- HMM algorithms
  - Likelihood computation (forward algorithm)
  - Most probable state sequence (Viterbi algorithm)
  - Estimting the parameters (EM algorithm)

# Fundamental Equation of Statistical Speech Recognition

If $\mathbf{X}$ is the sequence of acoustic feature vectors (observations) and $\mathbf{W}$ denotes a word sequence, the most likely word sequence $\mathbf{W}^*$ is given by

$$\mathbf{W}^* = \arg\max_{\mathbf{W}} P(\mathbf{W} \mid \mathbf{X})$$

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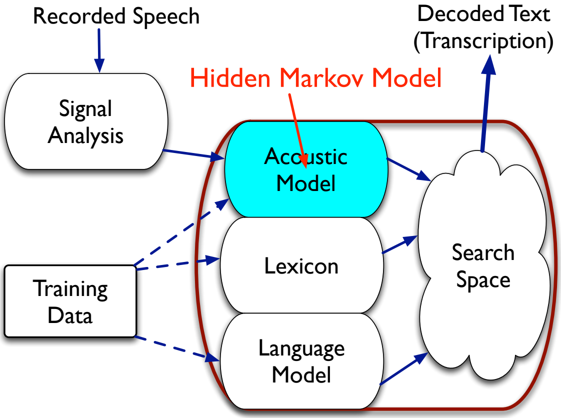
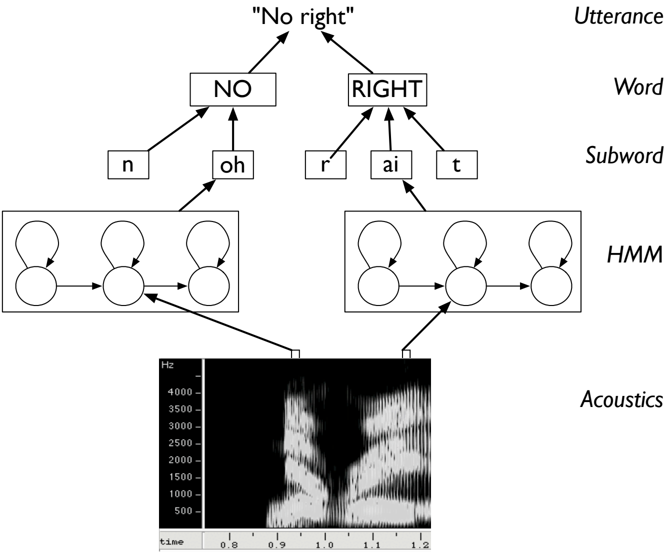
Applying Bayes' Theorem:

$$\begin{aligned} P(\mathbf{W} \mid \mathbf{X}) &= \frac{p(\mathbf{X} \mid \mathbf{W}) P(\mathbf{W})}{p(\mathbf{X})} \\ &\propto p(\mathbf{X} \mid \mathbf{W}) P(\mathbf{W}) \\ \mathbf{W}^* &= \arg\max_{\mathbf{W}} \underbrace{p(\mathbf{X} \mid \mathbf{W})}_{\substack{\text{Acoustic} \\ \text{model}}} \quad \underbrace{P(\mathbf{W})}_{\substack{\text{Language} \\ \text{model}}} \end{aligned}$$

# Acoustic Modelling



# Hierarchical modelling of speech



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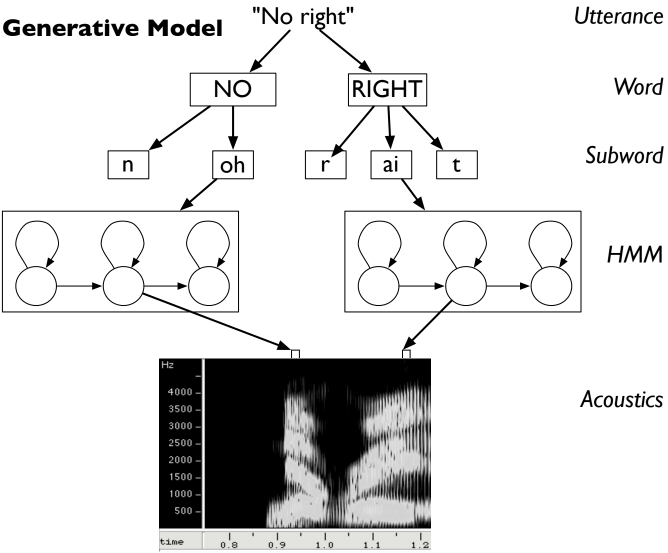
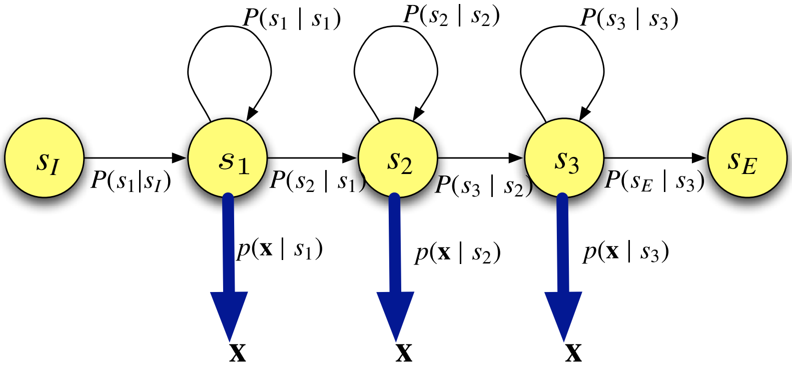


# Acoustic Model: Continuous Density HMM




Probabilistic finite state automaton

Paramaters $\boldsymbol{\lambda}$:

- Transition probabilities: $a_{kj} = P(S = j \mid S = k)$
- Output probability density function: $b_j(\mathbf{x}) = p(\mathbf{x} \mid S = j)$

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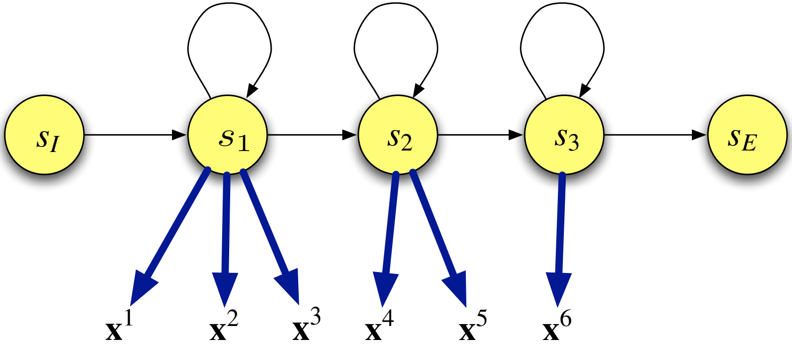


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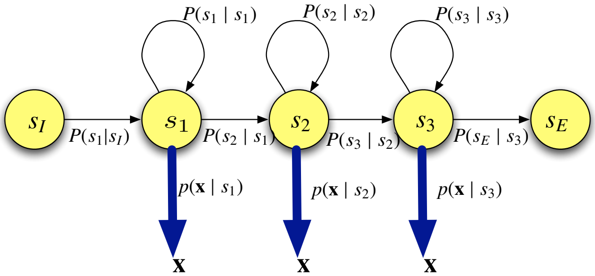


1. **Observation independence** An acoustic observation **x** is conditionally independent of all other observations given the state that generated it
2. **Markov process** A state is conditionally independent of all other states given the previous state

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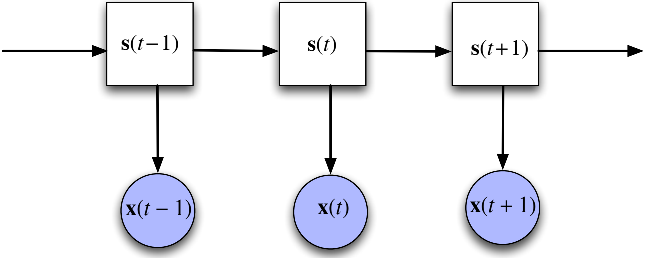


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# HMM OUTPUT DISTRIBUTION

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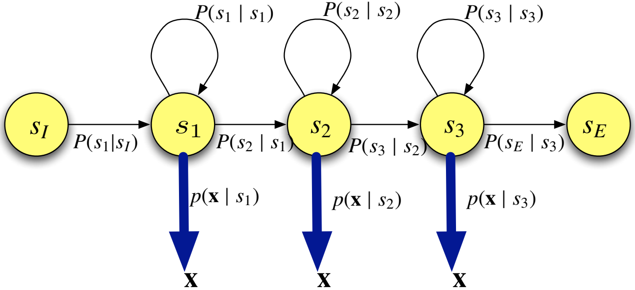

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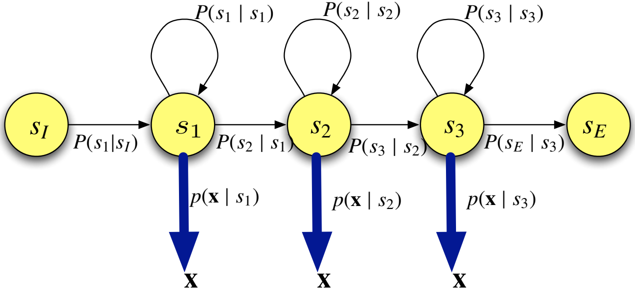


Single multivariate Gaussian with mean $\boldsymbol{\mu}_j$, covariance matrix $\boldsymbol{\Sigma}_j$:

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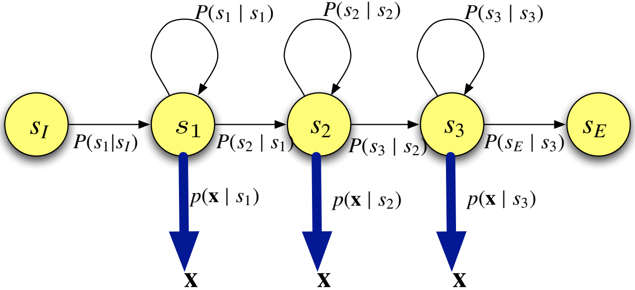


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$M$-component Gaussian mixture model:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid j) = \sum_{m=1}^{M} c_{jm} \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_{jm}, \boldsymbol{\Sigma}_{jm})$$

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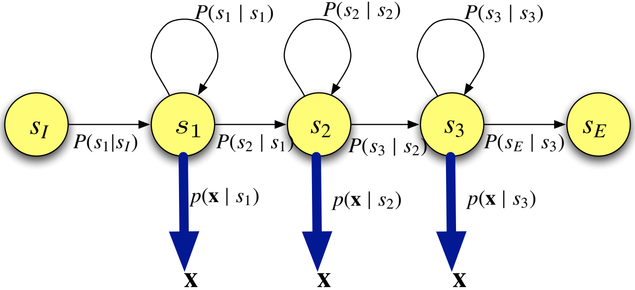


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# Background: cdf

Consider a real valued random variable $X$

- Cumulative distribution function (cdf) $F(x)$ for $X$:

$$F(x) = P(X \leq x)$$

- To obtain the probability of falling in an interval we can do the following:

$$\begin{aligned} P(a < X \leq b) &= P(X \leq b) - P(X \leq a) \\ &= F(b) - F(a) \end{aligned}$$

# Background: pdf

- The rate of change of the cdf gives us the *probability density function* (pdf), $p(x)$:

$$p(x) = \frac{d}{dx}F(x) = F'(x)$$
$$F(x) = \int_{-\infty}^{x} p(x)\mathrm{d}x$$

- $p(x)$ is **not** the probability that $X$ has value $x$. But the pdf is proportional to the probability that $X$ lies in a small interval centred on $x$.
- Notation: $p$ for pdf, $P$ for probability

# The Gaussian distribution (univariate)

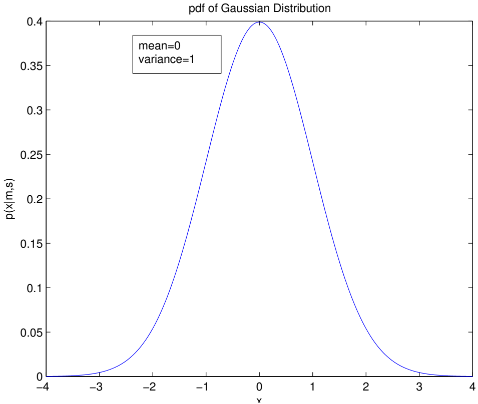
- The Gaussian (or Normal) distribution is the most common (and easily analysed) continuous distribution
- It is also a reasonable model in many situations (the famous "bell curve")
- If a (scalar) variable has a Gaussian distribution, then it has a probability density function with this form:

$$p(x|\mu, \sigma^2) = N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- The Gaussian is described by two parameters:
  - the mean $\mu$ (location)
  - the variance $\sigma^2$ (dispersion)

# Plot of Gaussian distribution

- Gaussians have the same shape, with the location controlled by the mean, and the spread controlled by the variance
- One-dimensional Gaussian with zero mean and unit variance ($\mu = 0$, $\sigma^2 = 1$):



# Properties of the Gaussian distribution

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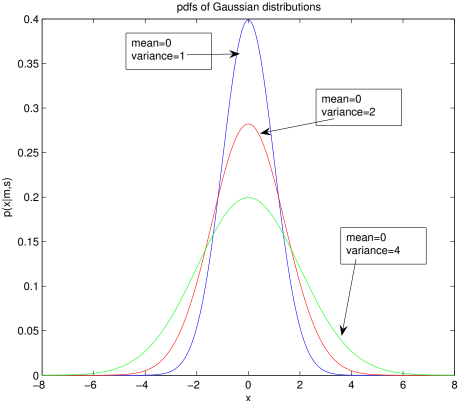

# Parameter estimation

- Estimate mean and variance parameters of a Gaussian from data $x^{(1)}, x^{(2)}, \ldots, x^{(N)}$
- Use sample mean and sample variance estimates:

$$\mu = \frac{1}{N} \sum_{n=1}^{N} x^{(n)} \qquad \text{(sample mean)}$$

$$\sigma^2 = \frac{1}{N} \sum_{n=1}^{N} (x^{(n)} - \mu)^2 \qquad \text{(sample variance)}$$

# Exercise — maximum likelihood estimation (MLE)

Consider the log likelihood of a set of $N$ training data points $\{x^{(1)}, \ldots, x^{(N)}\}$ being generated by a Gaussian with mean $\mu$ and variance $\sigma^2$:

$$L = \ln p(\{x^{(1)}, \ldots, x^{(n)}\} \mid \mu, \sigma^2) = -\frac{1}{2} \sum_{n=1}^{N} \left( \frac{(x^{(n)} - \mu)^2}{\sigma^2} - \ln \sigma^2 - \ln(2\pi) \right)$$

$$= -\frac{1}{2\sigma^2} \sum_{n=1}^{N} (x^{(n)} - \mu)^2 - \frac{N}{2} \ln \sigma^2 - \frac{N}{2} \ln(2\pi)$$

By maximising the the log likelihood function with respect to $\mu$ show that the maximum likelihood estimate for the mean is indeed the sample mean:

$$\mu_{ML} = \frac{1}{N} \sum_{n=1}^{N} x^{(n)}.$$

# The multivariate Gaussian distribution

- The $d$-dimensional vector $\mathbf{x} = (x_1, \ldots, x_d)^T$ follows a multivariate Gaussian (or normal) distribution if it has a probability density function of the following form:

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

The pdf is parameterized by the mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)^T$ and the covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \ldots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \ldots & \sigma_{dd} \end{pmatrix}$.

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- The 1-dimensional Gaussian is a special case of this pdf
- The argument to the exponential $0.5(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is referred to as a *quadratic form*.

# Covariance matrix

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- $\boldsymbol{\Sigma}$ is a $d \times d$ symmetric matrix:

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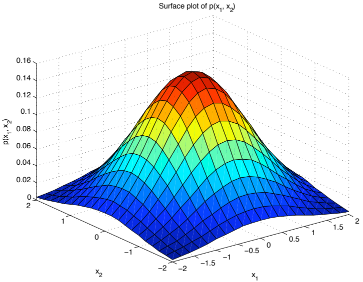
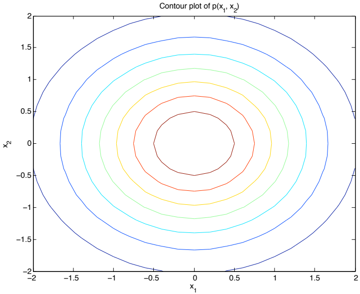
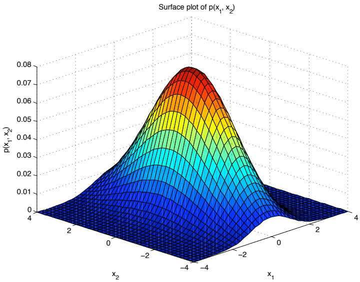
# Spherical Gaussian

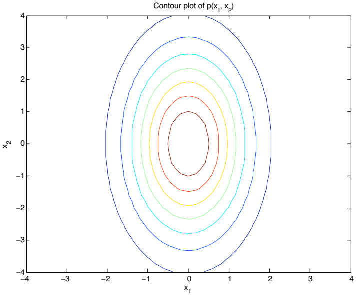



$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_{12} = 0$$

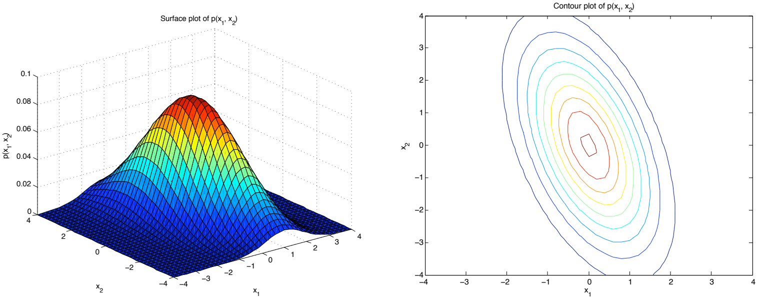
# Diagonal Covariance Gaussian




$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad \rho_{12} = 0$$

# Full covariance Gaussian




$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \qquad \rho_{12} = -0.5$$

# Parameter estimation of a multivariate Gaussian distribution

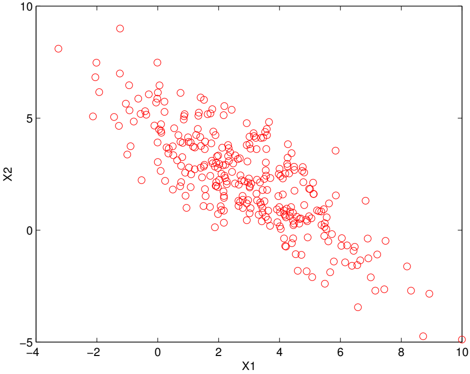
- It is possible to show that the mean vector $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}$ that maximize the likelihood of the training data are given by:

$$\hat{\boldsymbol{\mu}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}^{(n)}$$

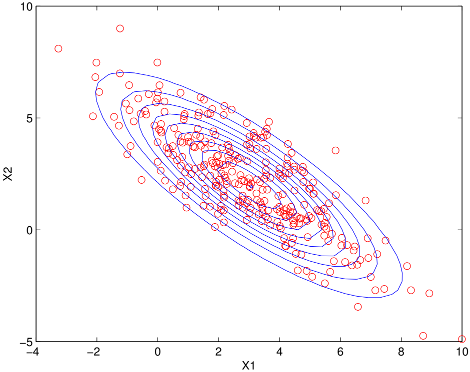
$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{n=1}^{N} (\mathbf{x}^{(n)} - \hat{\boldsymbol{\mu}})(\mathbf{x}^{(n)} - \hat{\boldsymbol{\mu}})^T$$

- The mean of the distribution is estimated by the sample mean and the covariance by the sample covariance

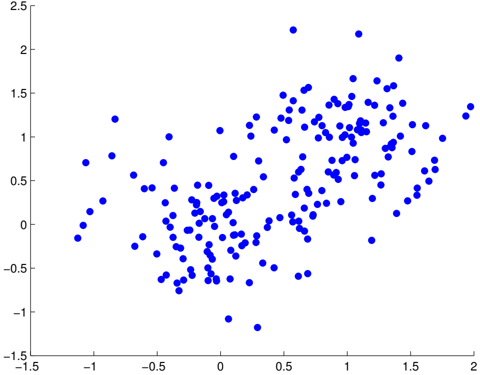
# Example data



# Maximum likelihood fit to a Gaussian

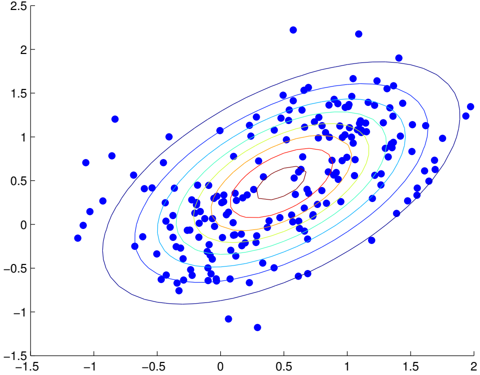


# Data in clusters (example 1)




$$\boldsymbol{\mu}_1 = [0 \quad 0]^T \qquad \boldsymbol{\mu}_2 = [1 \quad 1]^T \qquad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = 0.2\,\mathbf{I}$$

# Example 1 fit by a Gaussian

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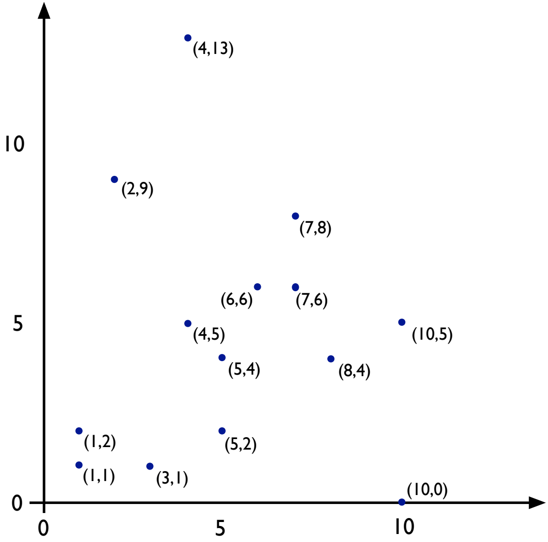
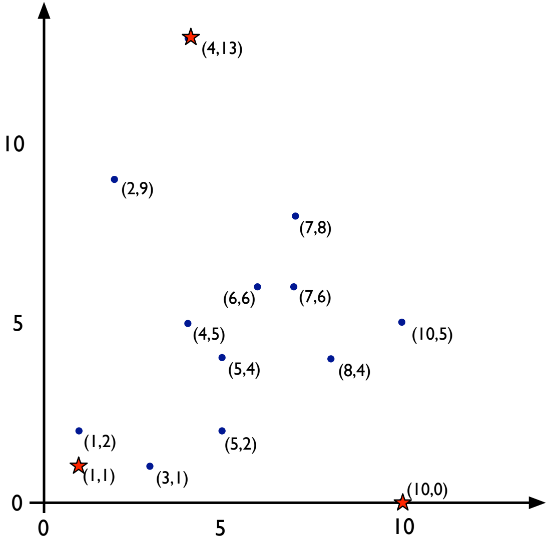
# k-means clustering

- k-means is an automatic procedure for clustering unlabelled data
- Requires a prespecified number of clusters
- Clustering algorithm chooses a set of clusters with the minimum within-cluster variance
- Guaranteed to converge (eventually)
- Clustering solution is dependent on the initialisation

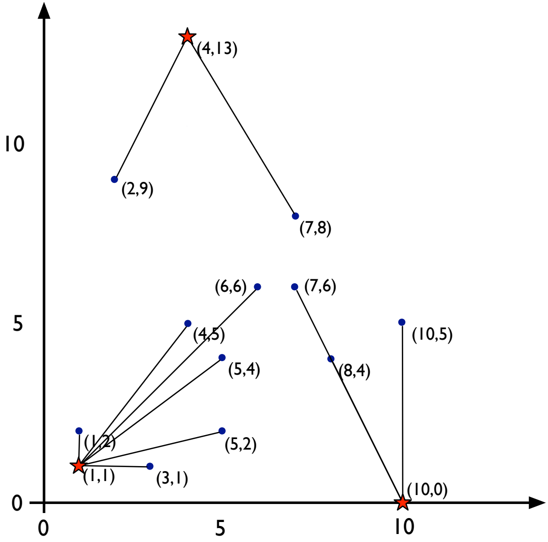
# k-means example: data set



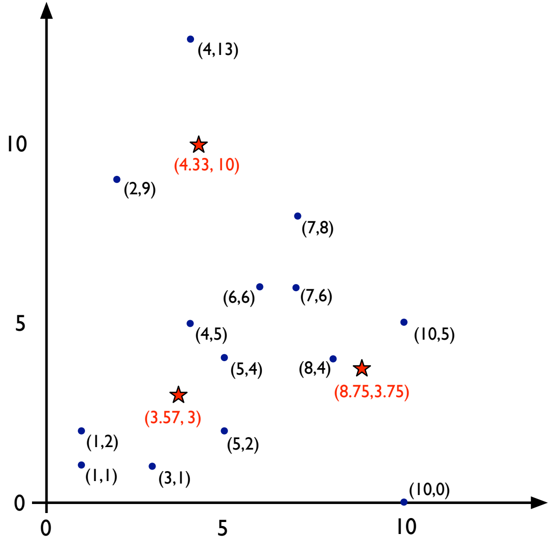
# k-means example: initialization



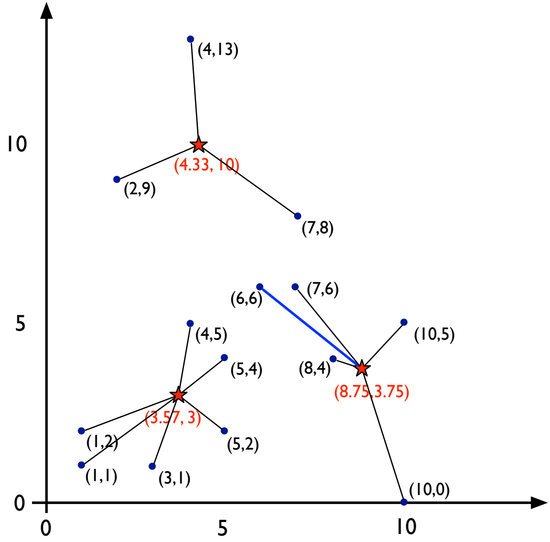
# k-means example: iteration 1 (assign points to clusters)



# k-means example: iteration 1 (recompute centres)



# k-means example: iteration 2 (assign points to clusters)



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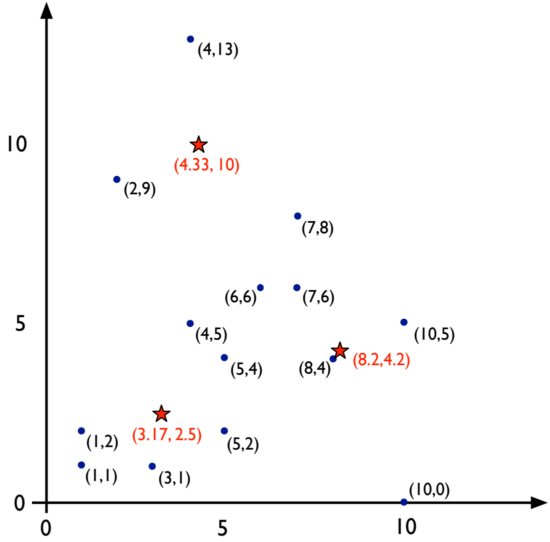
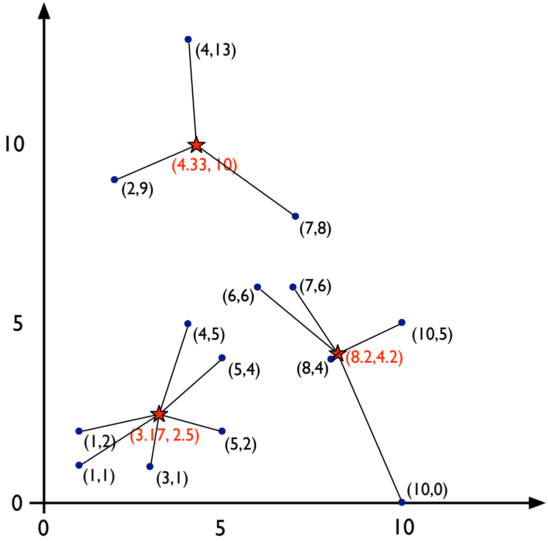


# k-means example: iteration 3 (assign points to clusters)




No changes, so converged

# Mixture model

- A more flexible form of density estimation is made up of a linear combination of component densities:

$$p(\mathbf{x}) = \sum_{j=1}^{M} p(\mathbf{x}|j)P(j)$$

- This is called a *mixture model* or a *mixture density*
- $p(\mathbf{x}|j)$: component densities
- $P(j)$: mixing parameters
- Generative model:
  1. Choose a mixture component based on $P(j)$
  2. Generate a data point $\mathbf{x}$ from the chosen component using $p(\mathbf{x}|j)$

# Component occupation probability

- We can apply Bayes' theorem:

$$P(j\,|\mathbf{x}) = \frac{p(\mathbf{x}|\,j\,)P(\,j\,)}{p(\mathbf{x})} = \frac{p(\mathbf{x}|\,j\,)P(\,j\,)}{\sum_{j=1}^{M} p(\mathbf{x}|\,j\,)P(\,j\,)}$$

- The posterior probabilities $P(\,j\,|\mathbf{x})$ give the probability that component $j$ was responsible for generating data point $\mathbf{x}$
- The $P(\,j\,|\mathbf{x})$s are called the *component occupation probabilities* (or sometimes called the *responsibilities*)
- Since they are posterior probabilities:

$$\sum_{j=1}^{M} P(\,j\,|\mathbf{x}) = 1$$

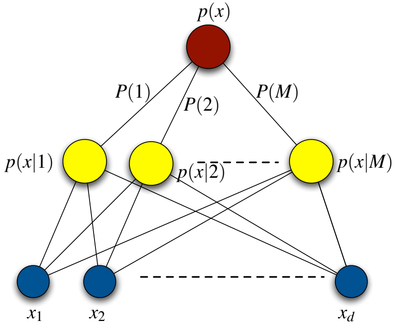
# Parameter estimation

- *If* we knew which mixture component was responsible for a data point:
  - we would be able to assign each point unambiguously to a mixture component
  - and we could estimate the mean for each component Gaussian as the sample mean (just like k-means clustering)
  - and we could estimate the covariance as the sample covariance
- *But* we don't know which mixture component a data point comes from...
- Maybe we could use the component occupation probabilities $P(j \mid \mathbf{x})$?

# Gaussian mixture model

- The most important mixture model is the *Gaussian Mixture Model* (GMM), where the component densities are Gaussians
- Consider a GMM, where each component Gaussian $N_j(\mathbf{x}; \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$ has mean $\boldsymbol{\mu}_j$ and a spherical covariance $\boldsymbol{\Sigma}_j = \sigma_j^2 \mathbf{I}$

$$p(\mathbf{x}) = \sum_{j=1}^{M} P(j) p(\mathbf{x}|j) = \sum_{j=1}^{M} P(j) N_j(\mathbf{x}; \boldsymbol{\mu}_j, \sigma_j^2 \mathbf{I})$$



# GMM Parameter estimation when we know which component generated the data

- Define the indicator variable $z_{jn} = 1$ if component $j$ generated component $\mathbf{x}^{(n)}$ (and 0 otherwise)
- If $z_{jn}$ wasn't hidden then we could count the number of observed data points generated by $j$:

$$N_j = \sum_{n=1}^{N} z_{jn}$$

- And estimate the mean, variance and mixing parameters as:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_n z_{jn}\mathbf{x}^{(n)}}{N_j}$$

$$\hat{\sigma}_j^2 = \frac{\sum_n z_{jn}||\mathbf{x}^{(n)} - \hat{\boldsymbol{\mu}}_j||^2}{N_j}$$

$$\hat{P}(j) = \frac{1}{N}\sum_n z_{jn} = \frac{N_j}{N}$$

# Soft assignment

- Estimate "soft counts" based on the component occupation probabilities $P(j\,|\mathbf{x}^{(n)})$:

$$N_j^* = \sum_{n=1}^{N} P(j\,|\mathbf{x}^{(n)})$$

- We can imagine assigning data points to component $j$ weighted by the component occupation probability $P(j\,|\mathbf{x}^{(n)})$
- So we could imagine estimating the mean, variance and prior probabilities as:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_n P(j\,|\mathbf{x}^{(n)})\mathbf{x}^{(n)}}{\sum_n P(j\,|\mathbf{x}^{(n)})} = \frac{\sum_n P(j\,|\mathbf{x}^{(n)})\mathbf{x}^{(n)}}{N_j^*}$$

$$\hat{\sigma}_j^2 = \frac{\sum_n P(j\,|\mathbf{x}^{(n)})||\mathbf{x}^{(n)} - \boldsymbol{\mu}_j||^2}{\sum_n P(j\,|\mathbf{x}^{(n)})} = \frac{\sum_n P(j\,|\mathbf{x}^{(n)})||\mathbf{x}^{(n)} - \boldsymbol{\mu}_j||^2}{N_j^*}$$

$$\hat{P}(j) = \frac{1}{N}\sum_n P(j\,|\mathbf{x}^{(n)}) = \frac{N_j^*}{N}$$

# EM algorithm

- *Problem!* Recall that:

$$P(j\,|\mathbf{x}) = \frac{p(\mathbf{x}|\,j\,)P(\,j\,)}{p(\mathbf{x})} = \frac{p(\mathbf{x}|\,j\,)P(\,j\,)}{\sum_{j=1}^{M} p(\mathbf{x}|\,j\,)P(\,j\,)}$$

  We need to know $p(\mathbf{x}|\,j\,)$ and $P(\,j\,)$ to estimate the parameters of $p(\,j\,|\mathbf{x})$, and to estimate $P(\,j\,)$....
- Solution: an iterative algorithm where each iteration has two parts:
  - Compute the component occupation probabilities $P(\,j\,|\mathbf{x})$ using the current estimates of the GMM parameters (means, variances, mixing parameters) (E-step)
  - Computer the GMM parameters using the current estimates of the component occupation probabilities (M-step)
- Starting from some initialization (e.g. using k-means for the means) these steps are alternated until convergence
- This is called the *EM Algorithm* and can be shown to maximize the likelihood

# Maximum likelihood parameter estimation

- The likelihood of a data set $\mathbf{X} = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}\}$ is given by:

$$\mathcal{L} = \prod_{n=1}^{N} p(\mathbf{x}^{(n)}) = \prod_{n=1}^{N} \sum_{j=1}^{M} p(\mathbf{x}^{(n)} | j) P(j)$$

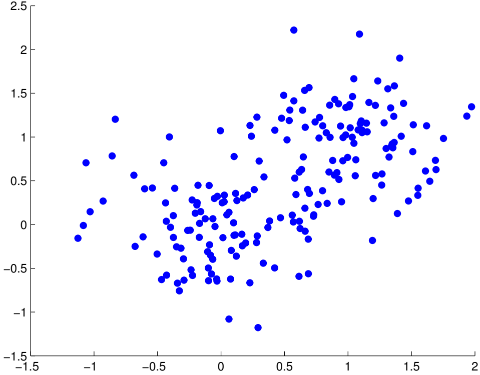
- We can regard the *negative log likelihood* as an error function:

$$E = -\ln \mathcal{L} = -\sum_{n=1}^{N} \ln p(\mathbf{x}^{(n)})$$

$$= -\sum_{n=1}^{N} \ln \left( \sum_{j=1}^{M} p(\mathbf{x}^{(n)} | j) P(j) \right)$$

- Considering the derivatives of $E$ with respect to the parameters, gives expressions like the previous slide

# Example 1 fit using a GMM



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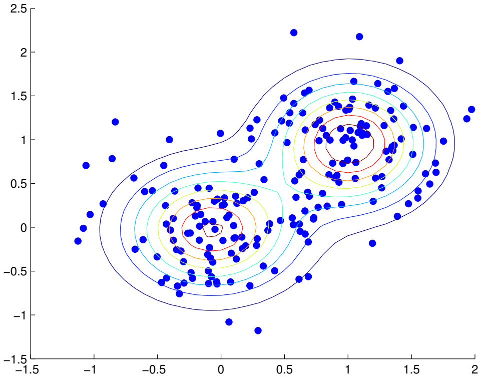
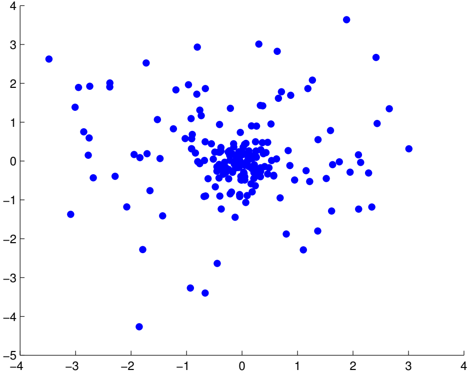


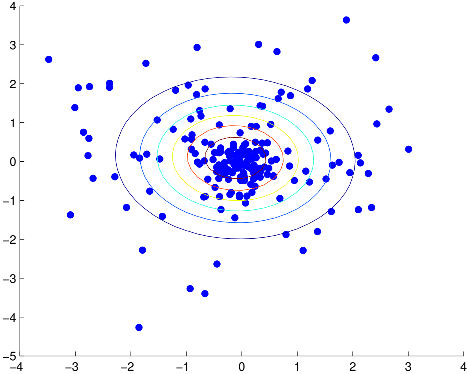

Fitted with a two component GMM using EM

# Peakily distributed data (Example 2)




$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = [0 \quad 0]^T \qquad \boldsymbol{\Sigma}_1 = 0.1\mathbf{I} \qquad \boldsymbol{\Sigma}_2 = 2\mathbf{I}$$

# Example 2 fit by a Gaussian

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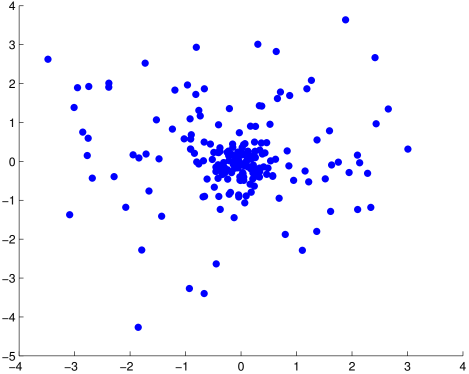

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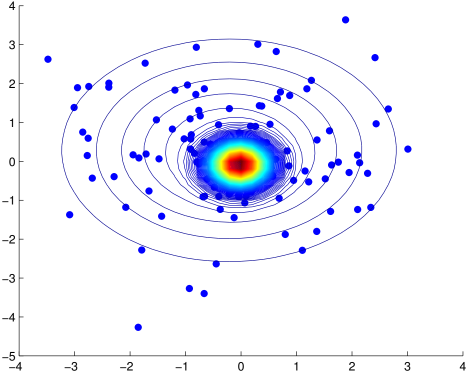


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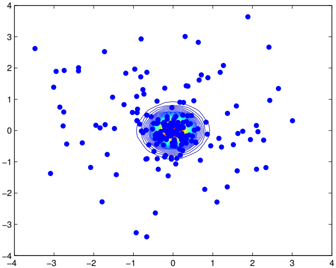
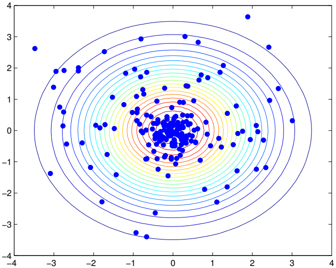
# Example 2: component Gaussians

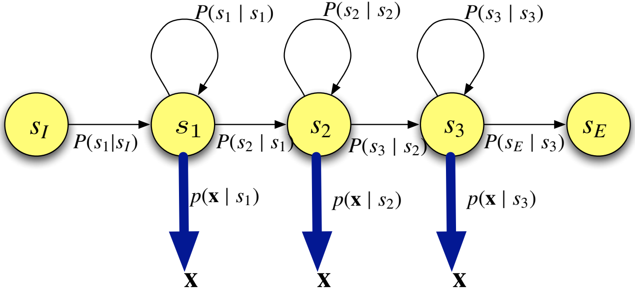


# Comments on GMMs

- GMMs trained using the EM algorithm are able to self organize to fit a data set
- Individual components take responsibility for parts of the data set (probabilistically)
- Soft assignment to components not hard assignment — "soft clustering"
- GMMs scale very well, e.g.: large speech recognition systems can have 30,000 GMMs, each with 32 components: sometimes 1 million Gaussian components!! And the parameters all estimated from (a lot of) data by EM

# Back to HMMs...




Output distribution:

- Single multivariate Gaussian with mean $\boldsymbol{\mu}_j$, covariance matrix $\boldsymbol{\Sigma}_j$:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid S = j) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$$

- $M$-component Gaussian mixture model:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid S = j) = \sum_{m=1}^{M} c_{jm} \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_{jm}, \boldsymbol{\Sigma}_{jm})$$

# The three problems of HMMs

Working with HMMs requires the solution of three problems:

1. **Likelihood** Determine the overall likelihood of an observation sequence $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_t, \ldots, \mathbf{x}_T)$ being generated by an HMM. (NB: $\mathbf{x}_t$ is used to denote $\mathbf{x}^{(t)}$ hereafter)

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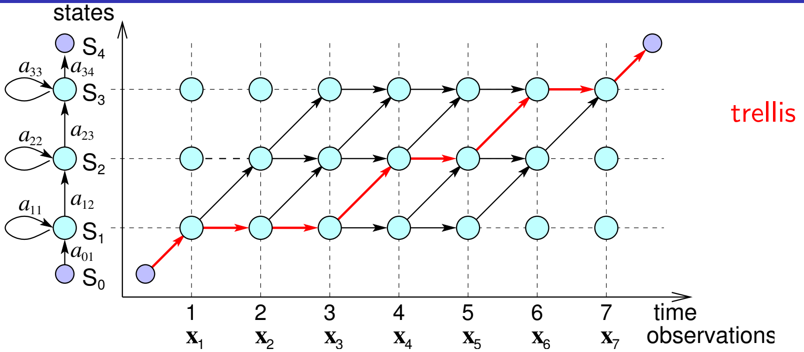
2. **Decoding** Given an observation sequence and an HMM, determine the most probable hidden state sequence
3. **Training** Given an observation sequence and an HMM, learn the best HMM parameters $\boldsymbol{\lambda} = \{\{a_{jk}\}, \{b_j()\}\}$

# 1. Likelihood: how to calculate?




$$
\begin{aligned}
P(X, \mathrm{path}_j | \Lambda) &= P(X | \mathrm{path}_j, \Lambda) P(\mathrm{path}_j | \Lambda) \\
&= P(X | s_0 s_1 s_1 s_1 s_2 s_2 s_3 s_3 s_4, \Lambda) P(s_0 s_1 s_1 s_1 s_2 s_2 s_3 s_3 s_4 | \Lambda) \\
&= b_1(\mathbf{x}_1) b_1(\mathbf{x}_2) b_1(\mathbf{x}_3) b_2(\mathbf{x}_4) b_2(\mathbf{x}_5) b_3(\mathbf{x}_6) b_3(\mathbf{x}_7) a_{01} a_{11} a_{11} a_{12} a_{22} a_{23} a_{33} a_{34}
\end{aligned}
$$

$$
P(X|\Lambda) = \sum_{\{\mathrm{path}_j\}} P(X, \mathrm{path}_j | \Lambda) \simeq \max_{\mathrm{path}_j} P(X, \mathrm{path}_j | \Lambda)
$$

forward(backward) algorithm — Viterbi algorithm

# 1. Likelihood: The Forward algorithm

- Goal: determine $p(\mathbf{X} \mid \boldsymbol{\lambda})$
- Sum over all possible state sequences $s_1 s_2 \ldots s_T$ that could result in the observation sequence $\mathbf{X}$

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- Hown many paths calculations in $p(\mathbf{X} \mid \boldsymbol{\lambda})$?

$$\sim \quad \underbrace{N \times N \times \cdots N}_{T \text{ times}} = N^T$$

$N$ : number of HMM states
$T$ : length of observation

e.g. $N = 3,\ T = 20 \ \rightarrow \approx\ 10^{10}$

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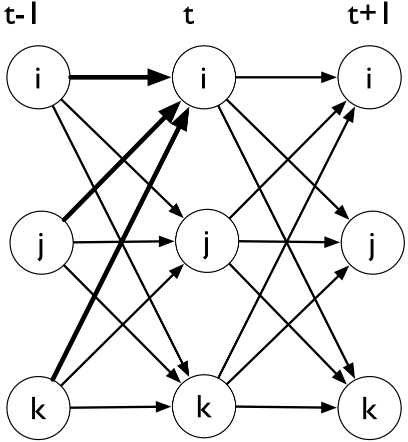
- Computation complexity for multiplication: $O(2T\,N^T)$
- The Forward algorithm reduces this to $O(TN^2)$

# Recursive algorithms on HMMs

Visualize the problem as a *state-time trellis*



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- Rather than enumerating each sequence, compute the probabilities recursively (exploiting the Markov assumption)
- *Forward probability*, $\alpha_t(j)$: the probability of observing the observation sequence $\mathbf{x}_1 \ldots \mathbf{x}_t$ and being in state $j$ at time $t$:

$$\alpha_t(j) = p(\mathbf{x}_1, \ldots, \mathbf{x}_t, S(t) = j \mid \boldsymbol{\lambda})$$

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- Initialization

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- Termination

$$p(\mathbf{X} \mid \boldsymbol{\lambda}) = \alpha_T(s_E) = \sum_{i=1}^{N} \alpha_T(i) a_{iE}$$

$s_I$: initial state, $s_E$: final state

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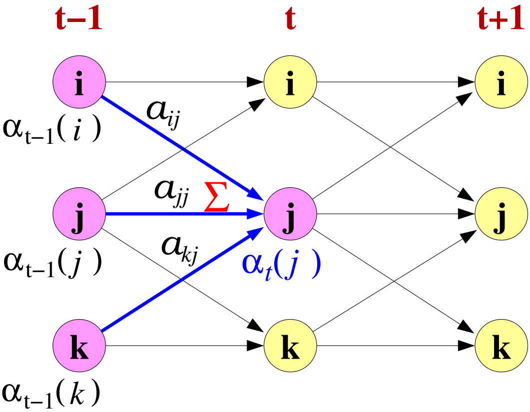

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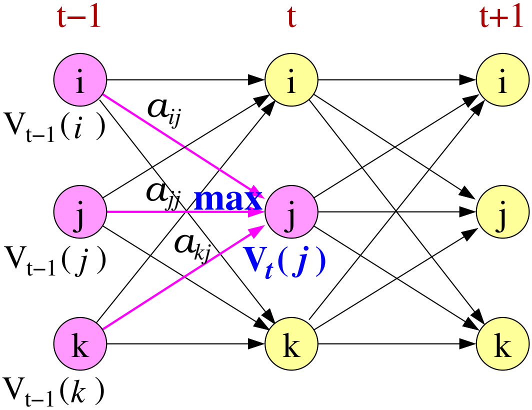
- Changing the recursion in this way gives the likelihood of the most probable path
- We need to keep track of the states that make up this path by keeping a sequence of *backpointers* to enable a Viterbi *backtrace*: the backpointer for each state at each time indicates the previous state on the most probable path

# Viterbi Recursion

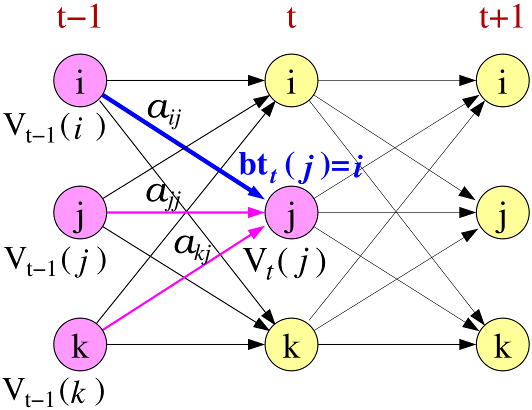
$$V_t(j) = \max_i V_{t-1}(i) a_{ij} b_j(\mathbf{x}_t)$$

Likelihood of the most probable path



# Viterbi Recursion

Backpointers to the previous state on the most probable path



# 2. Decoding: The Viterbi algorithm

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$$P^* = V_T(s_E) = \max_{i=1}^{N} V_T(i) a_{iE}$$
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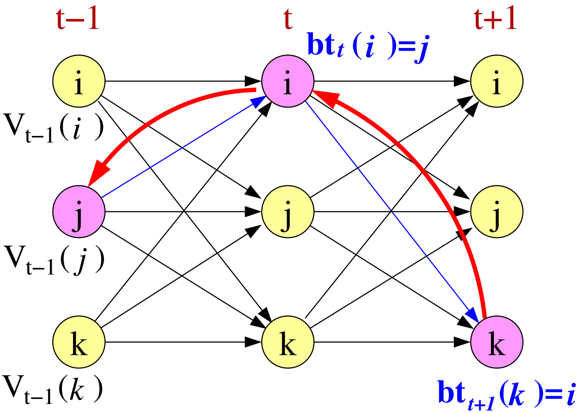
# Viterbi Backtrace

Backtrace to find the state sequence of the most probable path



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- Parameters $\boldsymbol{\lambda}$:
  - Transition probabilities $a_{ij}$:

$$\sum_j a_{ij} = 1$$

  - Gaussian parameters for state $j$:
    mean vector $\boldsymbol{\mu}_j$; covariance matrix $\boldsymbol{\Sigma}_j$

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- Likewise if $Z_j$ is the set of observed acoustic feature vectors assigned to state $j$, we can use the standard maximum likelihood estimates for the mean and the covariance:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_{\mathbf{x}\in Z_j} \mathbf{x}}{|Z_j|}$$

$$\hat{\boldsymbol{\Sigma}}_j = \frac{\sum_{\mathbf{x}\in Z_j} (\mathbf{x} - \hat{\boldsymbol{\mu}}_j)(\mathbf{x} - \hat{\boldsymbol{\mu}}_j)^T}{|Z_j|}$$

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  **E-step** estimate the state occupation probabilities (Expectation)

  **M-step** re-estimate the HMM parameters based on the estimated state occupation probabilities (Maximisation)

# Backward probabilities

- To estimate the state occupation probabilities it is useful to define (recursively) another set of probabilities—the *Backward* probabilities

  $$\beta_t(j) = p(\mathbf{x}_{t+1}, \ldots, \mathbf{x}_T \mid S(t) = j, \boldsymbol{\lambda})$$

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    $$p(\mathbf{X} \mid \boldsymbol{\lambda}) = \beta_0(I) = \sum_{j=1}^{N} a_{Ij} b_j(\mathbf{x}_1) \beta_1(j) = \alpha_T(s_E)$$

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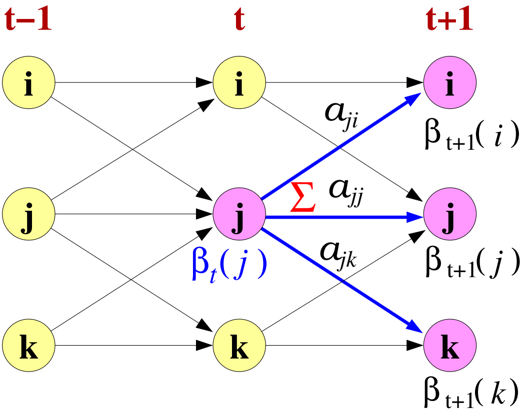

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- Since

$$\begin{aligned}
\alpha_t(j)\beta_t(j) &= p(\mathbf{x}_1, \ldots, \mathbf{x}_t, S(t) = j \mid \boldsymbol{\lambda}) \\
&\qquad p(\mathbf{x}_{t+1}, \ldots, \mathbf{x}_T \mid S(t) = j, \boldsymbol{\lambda}) \\
&= p(\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{x}_{t+1}, \ldots, \mathbf{x}_T, S(t) = j \mid \boldsymbol{\lambda}) \\
&= p(\mathbf{X}, S(t) = j \mid \boldsymbol{\lambda})
\end{aligned}$$

$$P(S(t) = j \mid \mathbf{X}, \boldsymbol{\lambda}) = \frac{p(\mathbf{X}, S(t) = j \mid \boldsymbol{\lambda})}{p(\mathbf{X}|\boldsymbol{\lambda})}$$

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- The sum of state occupation probabilities through time for a state, may be regarded as a "soft" count
- We can use this "soft" alignment to re-estimate the HMM parameters:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_{t=1}^{T} \gamma_t(j)\mathbf{x}_t}{\sum_{t=1}^{T} \gamma_t(j)}$$

$$\hat{\boldsymbol{\Sigma}}_j = \frac{\sum_{t=1}^{T} \gamma_t(j)(\mathbf{x}_t - \hat{\boldsymbol{\mu}}_j)(\mathbf{x} - \hat{\boldsymbol{\mu}}_j)^T}{\sum_{t=1}^{T} \gamma_t(j)}$$

# Re-estimation of transition probabilities

- Similarly to the state occupation probability, we can estimate $\xi_t(\, i\, ,\, j\,)$, the probability of being in $i$ at time $t$ and $j$ at $t+1$, given the observations:

$$\begin{aligned}
\xi_t(\, i\, ,\, j\,) &= P(S(t) = \, i\, , S(t+1) = \, j \, \mid \mathbf{X}, \boldsymbol{\lambda}) \\
&= \frac{P(S(t) = \, i\, , S(t+1) = \, j\, , \mathbf{X} \mid \boldsymbol{\lambda})}{p(\mathbf{X}|\Lambda)} \\
&= \frac{\alpha_t(\, i\,) a_{ij} b_j(\mathbf{x}_{t+1}) \beta_{t+1}(\, j\,)}{\alpha_T(s_E)}
\end{aligned}$$

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- We can use this to re-estimate the transition probabilities

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T} \xi_t(\,i\,,\,j\,)}{\sum_{k=1}^{N} \sum_{t=1}^{T} \xi_t(\,i\,,\,k\,)}$$

# Pulling it all together

- Iterative estimation of HMM parameters using the EM algorithm. At each iteration

  E step For all time-state pairs

  1. Recursively compute the forward probabilities $\alpha_t(j)$ and backward probabilities $\beta_t(j)$
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- The application of the EM algorithm to HMM training is sometimes called the Forward-Backward algorithm

# Extension to a corpus of utterances

- We usually train from a large corpus of $R$ utterances
- If $\mathbf{x}_t^r$ is the $t$th frame of the $r$th utterance $\mathbf{X}^r$ then we can compute the probabilities $\alpha_t^r(j)$, $\beta_t^r(j)$, $\gamma_t^r(j)$ and $\xi_t^r(i,j)$ as before
- The re-estimates are as before, except we must sum over the $R$ utterances, eg:

$$\hat{\boldsymbol{\mu}}_j = \frac{\sum_{r=1}^{R}\sum_{t=1}^{T}\gamma_t^r(j)\mathbf{x}_t^r}{\sum_{r=1}^{R}\sum_{t=1}^{T}\gamma_t^r(j)}$$

# Extension to Gaussian mixture model (GMM)

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- In this case an $M$-component Gaussian mixture model is an appropriate density function:

$$b_j(\mathbf{x}) = p(\mathbf{x} \mid j) = \sum_{m=1}^{M} c_{jm} \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_{jm}, \boldsymbol{\Sigma}_{jm})$$

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  Given enough components, this family of functions can model any distribution.

- Train using the EM algorithm, in which the component estimation probabilities are estimated in the E-step

# EM training of HMM/GMM

- Rather than estimating the state-time alignment, we estimate the component/state-time alignment, and component-state occupation probabilities $\gamma_t(j, m)$: the probability of occupying mixture component $m$ of state $j$ at time $t$

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- We can thus re-estimate the mean of mixture component $m$ of state $j$ as follows

$$\hat{\boldsymbol{\mu}}_{jm} = \frac{\sum_{t=1}^{T} \gamma_t(j, m)\mathbf{x}_t}{\sum_{t=1}^{T} \gamma_t(j, m)}$$

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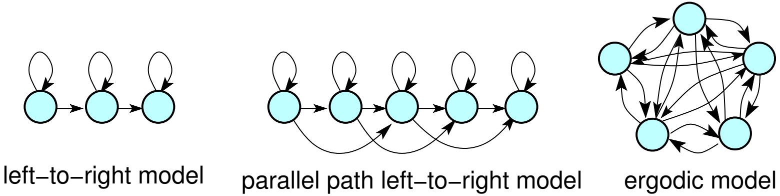
  And likewise for the covariance matrices (mixture models often use diagonal covariance matrices)
- The mixture coefficients are re-estimated in a similar way to transition probabilities:

$$\hat{c}_{jm} = \frac{\sum_{t=1}^{T} \gamma_t(j, m)}{\sum_{\ell=1}^{M} \sum_{t=1}^{T} \gamma_t(j, \ell)}$$

# Doing the computation

- The forward, backward and Viterbi recursions result in a long sequence of probabilities being multiplied
- This can cause floating point *underflow* problems
- In practice computations are performed in the log domain (in which multiplies become adds)
- Working in the log domain also avoids needing to perform the exponentiation when computing Gaussians

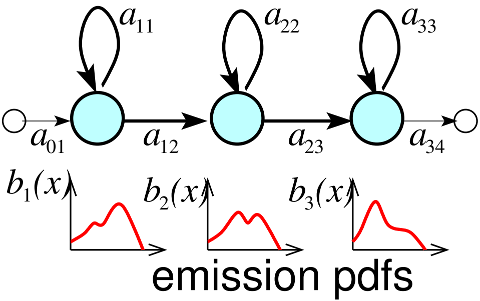
# A note on HMM topology




$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

Speech recognition: left-to-right HMM with $3 \sim 5$ states
Speaker recognition: ergodic HMM

# A note on HMM emission probabilities




| | Emission prob. | |
|---|---|---|
| Continuous (density) HMM | continuous density | GMM, NN/DNN |
| Discrete (probability) HMM | discrete probability | VQ |
| Semi-continuous HMM (tied-mixture HMM) | continuous density | tied mixture |

# Summary: HMMs

- HMMs provide a generative model for statistical speech recognition
- Three key problems
  1. Computing the overall likelihood: the Forward algorithm
  2. Decoding the most likely state sequence: the Viterbi algorithm
  3. Estimating the most likely parameters: the EM (Forward-Backward) algorithm
- Solutions to these problems are tractable due to the two key HMM assumptions
  1. Conditional independence of observations given the current state
  2. Markov assumption on the states

# References: HMMs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