Exercise 1

Recall from the lectures that for rewrite rules \( \text{L}_1 \Rightarrow \text{R}_1 \) and \( \text{L}_2 \Rightarrow \text{R}_2 \), a critical pair can be defined as:

\[
\langle \text{R}_1[\theta], \text{L}_1[\theta]/\{\text{R}_2[\theta]/s\} \rangle
\]

where \( \theta = \text{mgu} \) of \( s \) (subpart of \( \text{L}_1 \)) and \( \text{L}_2 \).

Now if we take:

\[
\begin{align*}
\text{L}_1 & : (u + v) + w \Rightarrow u + (v + w) \\
\text{R}_1 & : s(x) + y \Rightarrow x + s(y) \\
\text{L}_2 & : s(x) + y \Rightarrow x + s(y) \\
\text{R}_2 & : s(x) + y \Rightarrow x + s(y)
\end{align*}
\]

Then \( \theta = [s(x)/u, y/v] \), so the critical pair is given by

\[
\langle s(x) + (y + w), (x + s(y)) + w \rangle
\]

More concisely: The expression \( s(x) + y \) unifies with \( u + v \) with common instance \( s(x) + y \). \( s(x) + y + w \) can be rewritten to either \( (x + s(y)) + w \) or \( s(x) + (y + w) \). So the critical pair is:

\[
\langle s(x) + (y + w), (x + s(y)) + w \rangle
\]

Exercise 2

Two examples of critical pairs that are not joinable are: 1) starting from \( (X \cdot 0) \cdot Z \) we can get the critical pair \( \langle X \cdot (0 \cdot Z), X \cdot Z \rangle \); and 2) starting from \( (X \cdot i(X)) \cdot Z \) we can get the critical pair \( \langle 0 \cdot Z, X \cdot (i(X) \cdot Z) \rangle \). (2 marks for each critical pair). To get full marks, must mention that the critical pair is not joinable (or “conflatable”)
Exercise 3

1. An appropriate induction rule is:

\[
\begin{align*}
\Gamma \vdash P(\text{LEAF } x) & \quad \Gamma, P(x_1), P(x_2) \vdash P(\text{NODE } a \ x_1 \ x_2) \\
\Gamma \vdash \forall t. P(t)
\end{align*}
\]
2. An Isabelle definition (see theory file for proof):

```
primrec MIRROR :: "'a TREE => 'a TREE" where
  "MIRROR (LEAF x) = LEAF x"
| "MIRROR (NODE x l r) = NODE x (MIRROR r) (MIRROR l)"
```