Automated Reasoning

Coursework lecture: **Proving and Reasoning in Isabelle/HOL**

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Coursework overview

- Part 1: Propositional and first-order proofs [40%]
- Part 2: Geometry with order and signed areas [60%]

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- View them using `thm` *rule*.

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- Can you determine who is a knight and who is a knave?

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- In Isabelle we can use `case_tac`. E.g. `(case_tac "V x")`. We then have two subgoals: $\texttt{V x} \Longrightarrow \mathit{goal}$ and $\neg\, \texttt{V x} \Longrightarrow \mathit{goal}$.

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- We index the statement by n because one person may make more than one statement.
- We are also assuming that the domain of the quantifiers is all the inhabitants of the island (so you, as a visitor to the island, are not included).

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Knights and Knaves problems

- We can formalise the previous problem:
- `S 1 s = V z` and `S 1 z = K s ∧ K z`.
- Solution: Zippy cannot be a knight, because if what he said was true, then Sue would be telling a lie and then she is not a knight - contradiction. Hence Zippy is a knave, and as Sue is telling the truth, she is a knight.
- Suppose we get the formalisation wrong:
- `S 1 s = V z` and `S 1 z = K s` and `S 2 z = K z`.
- The previous analysis holds, thus Zippy is a knave, yet he makes a true statement (that Sue is a knight), so Zippy is a knight. Hence the problem is unsolvable.

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- A link to external provers: **sledgehammer**.

Reasoning with equality $(=)$

Rules:

$$\frac{t = s \qquad P\,s}{P\,t}\ \text{subst} \qquad\qquad \frac{s = t \qquad P\,s}{P\,t}\ \text{ssubst}$$

$$\frac{}{t = t}\ \text{refl} \qquad\qquad \frac{s = t}{t = s}\ \text{sym} \qquad\qquad \frac{r = s \qquad s = t}{r = t}\ \text{trans}$$

$$\frac{\forall x.\ f\,x = g\,x}{f = g}\ \text{ext}$$

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Are all of these rules necessary, or can some of them be derived from the others?

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Output:

1. $\forall c.\ \exists\, a\, b.\ a + 3 * b = c$

mult_zero_right: a * 0 = 0

add_0: 0 + a = a

add.commute: a + b = b + a

lemma "$\forall c :: \texttt{int}.\ \exists\, a\, b.\ a + 3 * b = c$"

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apply (rule add.commute)
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lemma "$\forall c :: \texttt{int}.\ \exists a\, b.\ a + 3 * b = c$"
  apply (rule allI)
  apply (rule_tac x = c in exI)
  apply (rule_tac x = 0 in exI)
  apply (subst mult_zero_right)
  apply (subst add.commute)
  apply (rule add_0)
  done

We can save all that variable instantiation using `subst`: rewriting.

But we will be using Isar:

Without subst:

```
lemma "∀c :: int. ∃a b. a + 3 * b = c"
proof
  fix c :: int
  have "c + 3 * 0 = c + 0"
    by (rule_tac s = 0 and
    t = "3 * 0" in ssubst,
    rule mult_zero_right, rule refl)
  also have "... = 0 + c"
    by (rule add.commute)
  also have "... = c" by (rule add_0)
  finally have "c + 3 * 0 = c"
    by (rule trans, rule_tac refl)
  then have "∃b. c + 3 * b = c"
    by (rule exI)
  then show "∃a b. a + 3 * b = c"
    by (rule exI)
qed
```

With subst:

```
lemma "∀c :: int. ∃a b. a + 3 * b = c"
proof
  fix c :: int
  have "c + 3 * 0 = c + 0"
    by (subst mult_zero_right,
    rule refl)
  also have "... = 0 + c"
    by (rule add.commute)
  also have "... = c" by (rule add_0)
  finally have "c + 3 * 0 = c"
    by (rule trans, rule_tac refl)
  then have "∃b. c + 3 * b = c"
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  then show "∃a b. a + 3 * b = c"
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Useful attributes to use with `subst`

`symmetric`: This swaps the left and right hand sides of the equality in *theorem*.

Usage: `subst` *theorem*`[symmetric]`

`asm`: This allows substitution into the assumption rather than the conclusion.

Usage: `subst(asm)` *theorem*

n, where n is a natural number: This allows substitution with the n^{th} occurrence in the goal of an expression that can be unified with the left-hand side of *theorem*.

Usage: `subst(`n`)` *theorem*

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lemma "$\forall c :: \texttt{int}.\ \exists\, a\, b.\ a + 3 * b = c$"
apply (rule allI)
apply (rule_tac x = c in exI)
apply (rule_tac x = 0 in exI)
apply (simp only: mult_zero_right add.commute add_0)
done

Method `simp` does substitution automatically (given the right rules!).

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  apply (rule_tac x = 0 in exI)
  apply simp
  done

Method `simp` does substitution automatically (given the right rules!).
...and the right rules are already in the Main library.

Isabelle's powerful tools

- **simp**: rewriting using equations.
  Uses:
  ```
  apply simp
  apply (simp add: eq_1 ... eq_n)
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- **auto**: rewriting + proof search (using classical logic).

  Uses:
  ```
  apply auto
  apply (auto simp add: eq1 ... eqn)
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  ```
- Others: **blast**, **fast**, **force**, **fastforce**, **safe**, **algebra**, **linarith**, **arith**, **presburger**, **meson**, **metis**.

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lemma "$inj\_on\ f\ A \Longrightarrow$
$\exists g.\ g' f' A \subseteq A \wedge (\forall a \in A.\ g(f\ a) = a) \wedge (\forall b \in A.\ f(g(f\ b)) = f\ b)$"
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by (metis order_refl the_inv_into_f_f the_inv_into_onto)

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- **try**: `try0` + `sledgehammer` + counterexample checkers!
- Use them just like `sledgehammer`.

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- Getting familiar with axiomatic systems.
- In particular, Isabelle's *locales*.
- We will define familiar geometric objects in terms of new concepts (order, signed area).
- It will help if we relate the formal statements to our geometric intuition.

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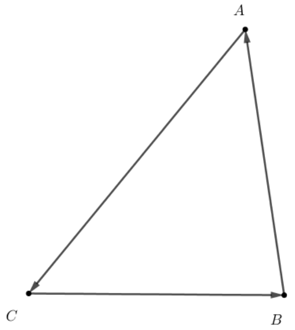
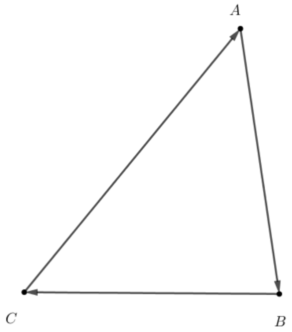
- We can interpret $\Delta\, x\, y\, z$ as the signed area of a triangle defined by the three arguments, x, y and z, of Δ.
- The signed area of a triangle is just the area of that triangle, multiplied by -1 if the points of that triangle are traversed clockwise, and by 1 otherwise.





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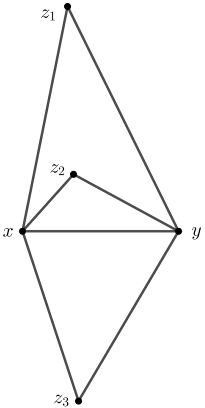

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- When in doubt *add brackets*.
- When in doubt *add type constraints*.
- During a proof, if you know your goal is unprovable (e.g., false), go back one step!
- Counterexample checkers (Quickcheck, Nitpick) can help you realise you made a wrong turn. Either call them directly (typing **quickcheck** or **nitpick**), or simply type **try**. (Especially important for knights and knaves).

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- Contact me by email: I.I.Morris@sms.ed.ac.uk.