Automated Reasoning

Lecture 18: Introduction to Binary Decision Diagrams (BDDs)

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based on originals by Paul Jackson diagrams from Huth & Ryan, LiCS, 2nd Ed.

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Recap

- Previously:
  - CTL and LTL Model Checking algorithms
- This time:
  - Binary Decision Diagrams
  - Reduced Binary Decision Diagrams
  - Reduced Ordered Binary Decision Diagrams
Model Checking needs Very Large Sets

Given a model $\mathcal{M} = \langle S, S_0, \rightarrow, L \rangle$ and a formula $\phi$, the CTL model checking algorithm translates CTL formulas into sets of states:

$$[\phi] \subseteq S$$

For realistic models, the size of $S$ can be enormous.
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Example: The NuSMV 2.5.4 distribution contains an example guidance, which is a model of (I think) part of the Shuttle’s autopilot. According to NuSMV:

```
NuSMV > print_reachable_states
######################################################################
system diameter: 70
reachable states: 2.10443e+14 (2^{47.5804}) out of 2.63684e+27 (2^{91.0909})
######################################################################
```

If each state is represented using 96 bits, it would need at least approx 2.52 petabytes to explicitly store the set of all reachable states.
Representing states as Boolean functions

Idea: represent sets of states as boolean functions.

1. Represent each state as a binary string in \( \{0, 1\}^k \)
2. Represent a set of states as a function \( f: \{0, 1\}^k \rightarrow \{0, 1\} \)
   \[ f(w) = 1 \text{ if the state represented by } w \text{ is in the set} \]
   \[ f(w) = 0 \text{ if the state represented by } w \text{ is not in the set} \]
⇒ representation of sets by their characteristic functions
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   \( \Rightarrow \) representation of sets by their characteristic functions

The set of all states could be represented as:

1. a data structure with \( 2.63684 \times 10^{27} \) nodes; or
2. the boolean function:
   \[ f(w) = 1 \]

How to represent boolean functions?
## Representations of Boolean Functions

From H&R, Figure 6.1

<table>
<thead>
<tr>
<th>Representation</th>
<th>compact?</th>
<th>test for satisf'y</th>
<th>validation</th>
<th>Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prop. Formulas</td>
<td>often</td>
<td>hard</td>
<td>hard</td>
<td>easy</td>
</tr>
<tr>
<td>Formulas in DNF</td>
<td>sometimes</td>
<td>easy</td>
<td>hard</td>
<td>easy</td>
</tr>
<tr>
<td>Formulas in CNF</td>
<td>sometimes</td>
<td>hard</td>
<td>easy</td>
<td>easy</td>
</tr>
<tr>
<td>Truth Tables</td>
<td>never</td>
<td>hard</td>
<td>hard</td>
<td>hard</td>
</tr>
<tr>
<td>Reduced OBDDs</td>
<td>often</td>
<td>easy</td>
<td>easy</td>
<td>medium</td>
</tr>
</tbody>
</table>

Easy/medium/hard refers to the time (or space) complexity in each case.

Note: With a truth table representation, while operations are conceptually easy, especially when table rows are always listed in some standard order, the time complexities are hard, as table sizes and hence operation time complexities are always exponential in the number of input variables.
Binary decision trees

Tree for the boolean function $f(x, y) = \neg x \land \neg y$

To compute value
1. Start at root
2. Take dashed line if value of var at current node is 0
3. Take solid line if value of var at current node is 1
4. Function value is value at terminal node reached
Binary decision diagram

Similar to Binary Decision Trees, except that nodes can have multiple in-edges.

A binary decision diagram (BDD) is:
a finite DAG (Directed Acyclic Graph) with:

▶ a unique initial node;
▶ all non-terminals labelled with a boolean variable;
▶ all terminals labelled with 0 or 1;
▶ all edges are labelled 0 (dashed) or 1 (solid);
▶ each non-terminal has exactly: one out-edge labelled 0, and one out-edge labelled 1.

We will use BDDs with two extra properties:

1. Reduced – eliminate redundancy
2. Ordered – canonical ordering of the boolean variables
Reducing BDDs I

remove duplicate nonterminals →

remove redundant test →
Reducing BDDs II

Removing duplicate non-terminals:
Reducing BDDs III

Removing redundant test:
Reduction Operations

1. **Removal of duplicate non-terminals.** If a BDD contains more than one terminal 0-node, then redirect all edges which point to such a 0-node to just one of them. Do the same with terminal nodes labelled 1.

2. **Removal of redundant tests.** If both outgoing edges of a node \( n \) point to the same node \( m \), then remove node \( n \), sending all its incoming edges to \( m \).

3. **Removal of duplicate non-terminals.** If two distinct nodes \( n \) and \( m \) in the BDD are the roots of structurally identical subBDDs, then eliminate one of them and redirect all its incoming edges to the other one.

All of these operations preserve the BDD-ness of the DAG.

A BDD is **reduced** if it has been simplified as much as possible using these reduction operations.
Generality of BDDs

Multiple occurrences of $x$

Different orderings on different paths
Ordered BDDs

- Let \([x_1, \ldots, x_n]\) be an ordered list of variables without duplicates;
- A BDD \(B\) has an ordering \([x_1, \ldots, x_n]\) if
  1. all variables of \(B\) occur in \([x_1, \ldots, x_n]\); and
  2. if \(x_j\) follows \(x_i\) on a path in \(B\) then \(j > i\)
- An ordered BDD (OBDD) is a BDD which has an ordering for some list of variables.
- The orderings of two OBDDs \(B\) and \(B'\) are compatible if there are no variables \(x, y\) such that
  - \(x\) is before \(y\) in the ordering for \(B\), and
  - \(y\) is before \(x\) in the ordering for \(B'\).

Theorem

For a given ordering, the reduced OBDD (ROBDD) representing a given function \(f\) is **unique**.

If \(B_1\) and \(B_2\) are two ROBDDs with compatible variable orderings representing the same boolean function, then they have identical structure.
Impact of variable ordering on size (I)

Consider the boolean function
\[ f(x_1, ..., x_{2n}) = (x_1 \lor x_2) \land (x_3 \lor x_4) \land ... \land (x_{2n-1} \lor x_{2n}) \]

With variable ordering \([x_1, x_2, x_3, ..., x_{2n}]\) ROBDD has \(2n + 2\) nodes

For \(n = 3\):
Impact of variable ordering on size (II)

With \([x_1, x_3, \ldots, x_{2n-1}, x_2, x_4, \ldots, x_{2n}]\) ROBDD has \(2^{n+1}\) nodes

For \(n = 3\):

There are various heuristics that can help with choosing orderings. However, improving a given ordering is NP-complete.
Impact of variable ordering on size III

Common ALU (Arithmetic Logic Unit) operations such as shifts, addition, subtraction, bitwise “and”, “or”, “exclusive or”, and xparity are all expressible using ROBDDs with total number of nodes linear in word size.

E.g., for even number of 1s for $n = 4$:

![Diagram](attachment:image.png)

No efficient ROBDD representation for multiply operation (they are all exponential size in the number of boolean variables).
Important of canonical representation

Having a canonical representation enables easy tests for

- **Absence of redundant variables.** A boolean function $f$ does not depend on an input variable $x$ if no nodes occur for $x$ in the ROBDD for $f$.

- **Semantic equivalence.** Check $f \equiv g$ by checking whether or not the ROBDDs for $f$ and $g$ have identical structure.

- **Validity.** Check if the BDD is identical to the one with just the terminal node $1$ and nothing else.

- **Satisfiability.** Check if the BDD is identical to the one with just the terminal node $0$ and nothing else.

- **Implication.** Check if $\forall \overrightarrow{x}. f(\overrightarrow{x}) \rightarrow g(\overrightarrow{x})$ by checking whether or not the ROBDD for $f \lor g$ is constant $0$. 
**Memoisation**

The representation of multiple ROBDDs can share memory.

Represent multiple BDDs using a large array of records:

<table>
<thead>
<tr>
<th>$v_0$</th>
<th>$l_0$</th>
<th>$h_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$l_1$</td>
<td>$h_1$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$l_2$</td>
<td>$h_2$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$l_3$</td>
<td>$h_3$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$l_4$</td>
<td>$h_4$</td>
</tr>
</tbody>
</table>

...  

| $v_{n-1}$ | $l_{n-1}$ | $h_{n-1}$ |

- each entry represents a BDD node
- $v_i$ is the variable label;
- $l_i$ is the index of the node pointed to for the false edge;
- $h_i$ is the index of the node pointed to for the true edge;
- Use fake $-1$ and $-2$ indexes to represent $0$ and $1$.
- Each ROBDD is represented by the index of its root.

Use a lookup table to ensure each entry is unique. So identical ROBBDs (and hence semantically equal functions) will have exactly the same index.
Summary

- BDDs (H&R 6.1)
  - Why BDDs?
  - Binary Decision Diagrams
  - Reduced Binary Decision Diagrams
  - Reduced Ordered Binary Decision Diagrams

- Next time:
  - Algorithms for implementing logical operations on BDDs
  - More details on implementing CTL MC with BDDs