Automated Reasoning

Lecture 15: Computation Tree Logic (CTL)

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based on originals by Paul Jackson

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Recap

- Previously:
  - *Linear-time* Temporal Logic

- This time:
  - A *branching-time* logic: Computation Tree Logic (CTL)
  - Syntax and Semantics
  - Comparison with LTL, CTL*
  - Model checking CTL
CTL Syntax

Assume a set $Atom$ of atom propositions.

\[ \phi, \psi ::= p \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \phi \rightarrow \psi \mid AX \phi \mid EX \phi \mid AF \phi \mid EF \phi \mid AG \phi \mid EG \phi \mid A[\phi U \psi] \mid E[\phi U \psi] \]

where $p \in Atom$.

Each temporal connective is a pair of a path quantifier:

- **A** — for all paths
- **E** — there exists a path

and an LTL-like temporal operator $X, F, G, U$.

Precedence (high-to-low): $(AX, EX, AF, EF, AG, EG, \neg), (\land, \lor), \rightarrow$
CTL Semantics 1: Transition Systems and Paths

(This is the same as for LTL)

Definition (Transition System)
A transition system \( \mathcal{M} = \langle S, \rightarrow, L \rangle \) consists of:

- \( S \) a finite set of states
- \( \rightarrow \subseteq S \times S \) transition relation
- \( L : S \rightarrow \mathcal{P}(\text{Atom}) \) a labelling function

such that \( \forall s_1. \exists s_2. s_1 \rightarrow s_2 \)

Definition (Path)
A path \( \pi \) in a transition system \( \mathcal{M} = \langle S, \rightarrow, L \rangle \) is an infinite sequence of states \( s_0, s_1, \ldots \) such that \( \forall i \geq 0. s_i \rightarrow s_{i+1} \).

Paths are written as: \( \pi = s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots \)
Satisfaction relation $\mathcal{M}$, $s \models \phi$ read as

*state s in model $\mathcal{M}$ satisfies CTL formula $\phi$*

We often leave $\mathcal{M}$ implicit.

The propositional connectives:

\[
\begin{align*}
  s & \models \top \\
  s & \not\models \bot \\
  s & \models p \quad \text{iff} \quad p \in L(s) \\
  s & \models \neg \phi \quad \text{iff} \quad s \not\models \phi \\
  s & \models \phi \land \psi \quad \text{iff} \quad s \models \phi \text{ and } s \models \psi \\
  s & \models \phi \lor \psi \quad \text{iff} \quad s \models \phi \text{ or } s \models \psi \\
  s & \models \phi \rightarrow \psi \quad \text{iff} \quad s \models \phi \text{ implies } s \models \psi
\end{align*}
\]
CTL Semantics 2: Satisfaction Relation

The temporal connectives:

\[ s |= AX \phi \iff \forall s'. (s \rightarrow s') \text{ implies } s' |= \phi \]

\[ s |= EX \phi \iff \exists s'. (s \rightarrow s') \text{ and } s' |= \phi \]

\[ s |= AG \phi \iff \forall \text{paths } \pi \text{ s.t. } \pi_o = s. \forall i. \pi_i |= \phi \]

\[ s |= EG \phi \iff \exists \text{path } \pi \text{ s.t. } \pi_o = s. \forall i. \pi_i |= \phi \]

\[ s |= AF \phi \iff \forall \text{paths } \pi \text{ s.t. } \pi_o = s. \exists i. \pi_i |= \phi \]

\[ s |= EF \phi \iff \exists \text{path } \pi \text{ s.t. } \pi_o = s. \exists i. \pi_i |= \phi \]

\[ s |= A[\phi U \psi] \iff \forall \text{paths } \pi \text{ s.t. } \pi_o = s. \exists i. \pi_i |= \psi \text{ and } \forall j < i. \pi_j |= \phi \]

\[ s |= E[\phi U \psi] \iff \exists \text{path } \pi \text{ s.t. } \pi_o = s. \exists i. \pi_i |= \psi \text{ and } \forall j < i. \pi_j |= \phi \]
For every next state, $\phi$ holds.
There exists a next state where $\phi$ holds.
For all paths, there exists a future state where $\phi$ holds.

$\text{AF } \phi$
There exists a path with a future state where $\phi$ holds.
AG $\phi$

For all paths, for all states along them, $\phi$ holds.
There exists a path such that, for all states along it, \( \phi \) holds.
For all paths, $\psi$ eventually holds, and $\phi$ holds at all states earlier.
E[ϕ U ψ]

Exists path where ψ eventually holds, and ϕ holds at all states earlier.
Examples of CTL formulas

- $\text{EF } \phi$
  
  there exists a future where eventually $\phi$ is true
Examples of CTL formulas

- **EF \( \phi \)**
  
  *there exists a future where eventually \( \phi \) is true*

- **AG AF \( \phi \)**
  
  *for all future states, \( \phi \) will eventually hold*
Examples of CTL formulas

- **EF ϕ**
  
  *there exists a future where eventually ϕ is true*

- **AG AF ϕ**
  
  *for all future states, ϕ will eventually hold*

- **AG (ϕ → AF ψ)**
  
  *for all future states, if ϕ holds, then ψ will eventually hold*
Examples of CTL formulas

- **EF** $\phi$
  
  *there exists a future where eventually $\phi$ is true*

- **AG AF** $\phi$
  
  *for all future states, $\phi$ will eventually hold*

- **AG ($\phi \rightarrow AF \psi$)**
  
  *for all future states, if $\phi$ holds, then $\psi$ will eventually hold*

- **AG ($\phi \rightarrow E[\phi U \psi]$)**
  
  *for all future states, if $\phi$ holds, then there is a future where $\psi$ eventually holds, and $\phi$ holds for all points in between*
Examples of CTL formulas

- **EF φ**
  
  *there exists a future where eventually φ is true*

- **AG AF φ**
  
  *for all future states, φ will eventually hold*

- **AG (φ → AF ψ)**
  
  *for all future states, if φ holds, then ψ will eventually hold*

- **AG (φ → E[φ U ψ])**
  
  *for all future states, if φ holds, then there is a future where ψ eventually holds, and φ holds for all points in between*

- **AG (φ → EG ψ)**
  
  *for all future states, if φ holds then there is a future where ψ always holds*
Examples of CTL formulas

- **EF $\phi$**
  
  *there exists a future where eventually $\phi$ is true*

- **AG AF $\phi$**
  
  *for all future states, $\phi$ will eventually hold*

- **AG ($\phi \rightarrow AF \psi$)**
  
  *for all future states, if $\phi$ holds, then $\psi$ will eventually hold*

- **AG ($\phi \rightarrow E[\phi U \psi]$)**
  
  *for all future states, if $\phi$ holds, then there is a future where $\psi$ eventually holds, and $\phi$ holds for all points in between*

- **AG ($\phi \rightarrow EG \psi$)** *for all future states, if $\phi$ holds then there is a future where $\psi$ always holds*

- **EF AG $\phi$**
  
  *there exists a possible state in the future, from where $\phi$ is always true*
CTL Equivalences

de Morgan dualities for the temporal connectives:

$$\neg \text{EX } \phi \equiv \text{AX } \neg \phi$$
$$\neg \text{EF } \phi \equiv \text{AG } \neg \phi$$
$$\neg \text{EG } \phi \equiv \text{AF } \neg \phi$$

Also have

$$\text{AF } \phi \equiv \text{A}[\top \mathbf{U} \phi]$$
$$\text{EF } \phi \equiv \text{E}[\top \mathbf{U} \phi]$$
$$\text{A}[\phi \mathbf{U} \psi] \equiv \neg(\text{E}[\neg \psi \mathbf{U} (\neg \phi \land \neg \psi)] \lor \text{EG } \neg \psi)$$

From these, one can show that the sets \{\text{AU, EU, EX}\} and \{\text{EU, EG, EX}\} are both adequate sets of temporal connectives.
Differences between LTL and CTL

LTL allows for questions of the form

- For all paths, does the LTL formula $\phi$ hold?
- Does there exist a path on which the LTL formula $\phi$ holds?  
  (Ask whether $\neg \phi$ holds on all paths, and ask for a counterexample)

CTL allows mixing of path quantifiers:

- $\text{AG} \ (p \rightarrow \text{EG} \ q)$
  
  For all paths, at every point, if $p$ is true, then there exists a path from there on which $q$ is always true.

However, some path properties are impossible to express in CTL

LTL: $\text{G} \ \text{F} \ p \rightarrow \text{G} \ \text{F} \ q$

CTL: $\text{AG} \ \text{AF} \ p \rightarrow \text{AG} \ \text{AF} \ q$

$\{ \}$ are not the same

Exist fair refinements of CTL that address this issue to some extent.

E.g., path quantifiers that only consider paths where something happens infinitely often.
CTL

\[
\begin{align*}
\text{LTL: } & \quad \text{G F } p \rightarrow \text{G F } q \\
\text{CTL: } & \quad \text{AG AF } p \rightarrow \text{AG AF } q
\end{align*}
\]
are not the same

LTL: for all paths \( \pi \), if \( p \) holds infinitely often on \( \pi \), then \( q \) holds infinitely often on \( \pi \)

CTL: if \( p \) holds infinitely often on all paths, then \( q \) holds infinitely often on all paths

Intuitively, in CTL we cannot fix a path \( \pi \) and talk about it.

CTL* addresses this by splitting formulas into state formulas \( \phi \) and path formulas \( \alpha \):

\[
\begin{align*}
\phi, \psi & ::= p \mid \neg \phi \mid \phi \land \psi \mid \phi \lor \psi \mid \phi \rightarrow \psi \\
& \quad \mid \text{A}[\alpha] \mid \text{E}[\alpha]
\end{align*}
\]

\[
\begin{align*}
\alpha, \beta & ::= \phi \mid \neg \alpha \mid \alpha \land \beta \mid \alpha \lor \beta \mid \alpha \rightarrow \beta \\
& \quad \mid \text{X} \alpha \mid \text{F} \alpha \mid \text{G} \alpha \mid \alpha \text{ U } \beta
\end{align*}
\]

Harder to model check.
CTL Model Checking

CTL Model Checking seeks to answer the question: *is it the case that*\
\[ \mathcal{M}, s \models \phi \]

*for all initial states* \( s \in S_0 \)?

CTL Model Checking algorithms usually fix \( \mathcal{M} = \langle S, \rightarrow, L \rangle \) and \( \phi \) and compute\
\[ \llbracket \phi \rrbracket_{\mathcal{M}} = \{ s \in S \mid \mathcal{M}, s \models \phi \} \]

“the denotation of \( \phi \) in the model \( \mathcal{M} \)”

The model checking question now becomes: \( S_0 \subseteq \llbracket \phi \rrbracket_{\mathcal{M}} \)?

*(The model \( \mathcal{M} \) is usually left implicit)*
Denotation Semantics for CTL

We compute $[\phi]$ recursively on the structure of $\phi$:

$[\top] = S$

$[\bot] = \emptyset$

$[p] = \{ s \in S \mid p \in L(s) \}$

$[\neg \phi] = S - [\phi]$

$[\phi \land \psi] = [\phi] \cap [\psi]$

$[\phi \lor \psi] = [\phi] \cup [\psi]$

$[\phi \rightarrow \psi] = (S - [\phi]) \cup [\psi]$

Since $[\phi]$ is always a finite set, these are computable.
Denotation Semantics of the Temporal Connectives

\[ [\text{EX } \phi] = \text{pre}_\exists([\phi]) \]

\[ [\text{AX } \phi] = \text{pre}_\forall([\phi]) \]

where

\[ \text{pre}_\exists(Y) = \{ s \in S \mid \exists s' \in S. (s \rightarrow s') \wedge s' \in Y \} \]

\[ \text{pre}_\forall(Y) = \{ s \in S \mid \forall s' \in S. (s \rightarrow s') \rightarrow s' \in Y \} \]

these are again computable, because \( Y \) and \( S \) are finite.

But what about the rest of the temporal connectives? e.g.

\[ [\text{EF } \phi] = \{ s \in S \mid \exists \text{path } \pi \text{ s.t. } s_0 = s. \exists i. \pi_i |= \phi \} \]

No obvious way to compute this: there are infinitely many paths \( \pi \)!
Approximating $[\text{EF } \phi]$ 

Define

$$\begin{align*}
\text{EF}_0 \phi &= \bot \\
\text{EF}_{i+1} \phi &= \phi \lor \text{EX} \text{EF}_i \phi
\end{align*}$$

Then

$$\begin{align*}
\text{EF}_1 \phi &= \phi \\
\text{EF}_2 \phi &= \phi \lor \text{EX} \phi \\
\text{EF}_3 \phi &= \phi \lor \text{EX} (\phi \lor \text{EX} \phi)
\end{align*}$$

$$\vdots$$

$s \in [\text{EF}_i \phi]$ if there exists a finite path of length $i - 1$ from $s$ and $\phi$ holds at some point along that path.

For a given model $\mathcal{M}$, let $n = |S|$. If there is a path of length $k > n$ on which $\phi$ holds somewhere, there will also be a path of length $n$. 
(Proof: take the $k$-length path and repeatedly cut out segments between repeated states.)

Therefore, for all $k > n$, $[\text{EF}_k \phi] = [\text{EF}_n \phi]$
Computing $[\text{EF } \phi]$ 

By a similar argument,

$$[\text{EF } \phi] = [\text{EF}_n \phi]$$

The approximations can be computed by recursion on $i$:

$$[\text{EF}_0 \phi] = \emptyset$$
$$[\text{EF}_{i+1} \phi] = \left[ \phi \right] \cup \text{pre}_{\exists}([\text{EF}_i \phi])$$

So we have an effective way of computing $[\text{EF } \phi]$. 
Approximating \([\text{EG } \phi]\)

Define

\[
\begin{align*}
\text{EG}_0 \phi & = \top \\
\text{EG}_{i+1} \phi & = \phi \land \text{EX} \text{EG}_i \phi
\end{align*}
\]

Then

\[
\begin{align*}
\text{EG}_1 \phi & = \phi \\
\text{EG}_2 \phi & = \phi \land \text{EX} \phi \\
\text{EG}_3 \phi & = \phi \land \text{EX} (\phi \land \text{EX} \phi)
\end{align*}
\]

\[\vdots\]

\[s \in [\text{EG}_i \phi]\] if there exists a finite path of length \(i - 1\) from \(s\) and \(\phi\) holds at every point along that path.

As with \([\text{EF } \phi]\), we have for all \(k > n\), \([\text{EG}_k \phi] = [\text{EG}_n \phi] = [\text{EG } \phi]\) and so we can compute \([\text{EG } \phi]\).
What’s happening here is that we are computing fixed points.

A set $X \subseteq S$ is a fixed point of a function $F : \mathcal{P}(S) \to \mathcal{P}(S)$ iff $F(X) = X$.

We have that

$$\lfloor EF_n \phi \rfloor \quad = \quad \lfloor EF_{n+1} \phi \rfloor$$
$$\quad = \quad \lfloor \phi \lor EX EF_n \phi \rfloor$$
$$\quad = \quad \lfloor \phi \rfloor \cup \text{pre}_\exists(\lfloor EF_n \phi \rfloor)$$

so $\lfloor EF_n \rfloor$ is a fixed point of $F(Y) = \lfloor \phi \rfloor \cup \text{pre}_\exists(Y)$.

Also, $\lfloor EF \phi \rfloor$ is a fixed point of $F$, since $\lfloor EF \phi \rfloor = \lfloor EF_n \phi \rfloor$.

More specifically, they are both the least fixed point of $F$. 
Fixed point Theorem

Let $F : \mathcal{P}(S) \to \mathcal{P}(S)$ be a function that takes sets to sets.

- $F$ is monotone iff $X \subseteq Y$ implies $F(X) \subseteq F(Y)$.
- Let $F^0(X) = X$ and $F^{i+1}(X) = F(F^i(X))$.
- Given a collection of sets $C \subseteq \mathcal{P}(S)$, a set $X \in C$ is
  1. the least element of $C$ if $\forall Y \in C. X \subseteq Y$; and
  2. the greatest element of $C$ if $\forall Y \in C. Y \subseteq X$.

Theorem (Knaster-Tarski (Special Case))

Let $S$ be a set with $n$ elements and $F : \mathcal{P}(S) \to \mathcal{P}(S)$ be a monotone function. Then

- $F^n(\emptyset)$ is the least fixed point of $F$; and
- $F^n(S)$ is the greatest fixed point of $F$.

(Proof: see H&R, Section 3.7.1)

This theorem justifies $F^n(\emptyset)$ and $F^n(S)$ being fixed points of $F$ without the need, as before, to appeal to further details about $F$. 
Denotational semantics of temporal connectives

When $F : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ is a monotone function, we write

- $\mu Y. F(Y)$ for the least fixed point of $F$; and
- $\nu Y. F(Y)$ for the greatest fixed point of $F$.

With this notation, we can define:

$$
\begin{align*}
\llbracket EF \phi \rrbracket &= \mu Y. \llbracket \phi \rrbracket \cup \text{pre}_\exists(Y) \\
\llbracket EG \phi \rrbracket &= \nu Y. \llbracket \phi \rrbracket \cap \text{pre}_\exists(Y) \\
\llbracket AF \phi \rrbracket &= \mu Y. \llbracket \phi \rrbracket \cup \text{pre}_\forall(Y) \\
\llbracket AG \phi \rrbracket &= \nu Y. \llbracket \phi \rrbracket \cap \text{pre}_\forall(Y) \\
\llbracket E[\phi U \psi] \rrbracket &= \mu Y. \llbracket \psi \rrbracket \cup (\llbracket \phi \rrbracket \cap \text{pre}_\exists(Y)) \\
\llbracket A[\phi U \psi] \rrbracket &= \mu Y. \llbracket \psi \rrbracket \cup (\llbracket \phi \rrbracket \cap \text{pre}_\forall(Y))
\end{align*}
$$

In every case, $F$ is monotone, so the Knaster-Tarski theorem assures us that the fixed point exists, and can be computed.
Further CTL Equivalences

The fixed point characterisations of the CTL temporal connectives justify some more equivalences between CTL formulas:

\[
\begin{align*}
\text{EF } \phi & \equiv \phi \lor \text{EX EF } \phi \\
\text{EG } \phi & \equiv \phi \land \text{EX EG } \phi \\
\text{AF } \phi & \equiv \phi \lor \text{AX AF } \phi \\
\text{AG } \phi & \equiv \phi \land \text{AX AG } \phi \\
\text{E}[\phi \ U \ \psi] & \equiv \psi \lor (\phi \land \text{EX E}[\phi \ U \ \psi]) \\
\text{A}[\phi \ U \ \psi] & \equiv \psi \lor (\phi \land \text{AX A}[\phi \ U \ \psi])
\end{align*}
\]
Summary

- CTL (H&R 3.4, 3.5, 3.6.1, 3.7)
  - CTL, Syntax and Semantics
  - Comparison with LTL, CTL*
  - Model Checking algorithm for CTL

- Next time:
  - (A taste of) The LTL Model Checking algorithm