Recap

- Last time: First-Order Logic
- This time: Representing mathematical concepts
Representing Knowledge

So far, we have:

▶ Seen the primitive rules of (first-order) logic
▶ Reasoned about abstract $P$s, $Q$s, and $R$s

But we usually want to reason in some mathematical theory. For example: number theory, real analysis, automata theory, euclidean geometry, ...

How do we represent this theory so we can prove theorems about it?

▶ Which logic do we use? — Propositional, FOL, Temporal, Hoare Logic, HOL?
▶ Do we axiomatise our theory, or define it in terms of more primitive concepts?
▶ What style do we use? e.g. functions vs. relations
Further Issues

What are the important theorems in our theory?

▶ Which formalisation is most useful?
▶ Is it easy to understand?
▶ Is it natural?
▶ How easy is it to reason with?

Often a matter of taste, or experience, or tradition, or efficiency of implementation, or following the idioms of the people you are working with. No single right way!

Granularity of the representation

▶ What primitives do we need? Consider geometry:
  ▶ Define lines in terms of points? (Tarski)
  ▶ Or take points and lines as primitive? (Hilbert)
▶ Or computing; should we treat programs as:
  ▶ State transition systems? (operational)
  ▶ Functions mapping inputs to outputs? (∼ denotational)
Axioms vs. Definitions

Let’s say we want to reason using the natural numbers \( \{0, 1, 2, 3, \ldots \} \)

**Axiomatise?** Assume a collection of function symbols and *unproven axioms*. For instance, the Peano axioms:

\[
\begin{align*}
\forall x. \neg (0 = S(x)) \\
\forall x. x + 0 &= x \\
\forall x. x + S(y) &= S(x + y)
\end{align*}
\]

\[\ldots\]

**Define?** If our logic has sets as a primitive (or are definable), then we can *define* the natural numbers via the von Neumann ordinals:

\[ 0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \ldots \]

Then we can *prove* the Peano axioms for this definition.
Axioms vs. Definitions

Axiomatisation:

▶ (+) Sometimes less work – finding a good definition, and (formally) working with it can be hard.
▶ (-) How do we know that our axiomatisation is adequate for our purposes, or is complete?
▶ (-) How do we know that our axiomatisation is consistent? Can we prove $\perp$ from our axioms (and hence everything)?

Definition:

▶ (-) Can be a lot of work, sometimes needing some ingenuity.
▶ (+++) If the underlying logic is consistent, then we are guaranteed to be consistent (c.f., “Why should you believe Isabelle” from Lecture 4). We have relative consistency.
Axiomatisation, an example: Set Theory

Let’s take FOL, a binary atomic predicate $\in$ and the following axiom for every formula $P$ with one free variable $x$:

$$\exists y. \forall x. x \in y \leftrightarrow P(x)$$

“For every predicate $P$ there is a set $y$ such that its members are exactly those that satisfy $P$”

We can now define empty set, pairing, union, intersection...
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$$y \in y \iff y \notin y$$

This is Russell’s paradox.
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Background: the axiom is called ”unrestricted comprehension”, it was replaced by:

$$\forall z. \exists y. \forall x. (x \in y \leftrightarrow (x \in z \land P(x)))$$

+ some other axioms to give ZF set theory.
Building up Definitions: Integers

Starting from the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \), we can define:

- each integer \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) as an **equivalence class** of pairs of natural numbers under the relation
  \[(a, b) \sim (c, d) \iff a + d = b + c;\]
- For example, \(-2\) is represented by the equivalence class \([0, 2)] = [(1, 3)] = [(100, 102)] = \ldots\.
- we define the sum and product of two integers as
  \[
  [(a, b)] + [(c, d)] = [(a + c, b + d)]
  [(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)];
  \]
- we define the set of **negative** integers as the set
  \[
  \{[(a, b)] \mid b > a\}.
  \]
- Exercise: show that the product of negative integers is non-negative.
Other Representation Examples

- The rationals $\mathbb{Q}$ can be defined as pairs of integers. Reasoning about the rationals therefore reduces to reasoning about the integers.
- The reals $\mathbb{R}$ can be defined as sets of rationals. Reasoning about the reals therefore reduces to reasoning about the rationals.
- The complex numbers $\mathbb{C}$ can be defined as pairs of reals. Reasoning about the complex numbers therefore reduces to reasoning about the reals.
- In this way, we have relative consistency.
  - If the theory of natural numbers is consistent, so is the theory of complex numbers.
Functions or Predicates?

We can represent some property $r$ holding between two objects $x$ and $y$ as:

- a function with equality $r(x) = y$
- a predicate $r(x, y)$

Is it better to use functions or predicates to represent properties?

It is not always clear which is best!
For example, suppose we represent division of real numbers (/) by a function \( \text{div} : \text{real} \times \text{real} \Rightarrow \text{real} \).

- We define \( \text{div}(x, y) \) when \( y \neq 0 \) in normal way
- What about division-by-zero? What is the value of \( \text{div}(x, 0) \)?
- In first-order logic, functions are assumed to be total, so we have to pick a value!
- We could choose a convenient element: say 0. That way:

\[
0 \leq x \rightarrow 0 \leq 1/x.
\]
Q) Can we represent division of real numbers (/) by a relation $\text{Div} : \text{real} \times \text{real} \times \text{real} \Rightarrow \text{bool}$ such that $\text{Div}(x, y, z)$ is

$\quad \triangleright \ x/y = z$ when $y \neq 0$, and
$\quad \triangleright \ \bot$ when $y = 0$?

A) Yes: $\text{Div}(x, y, z) \equiv x = y \times z \land \forall w. \ x = y \times w \rightarrow z = w$

That is, $z$ is that unique value such that $x = y \times z$.

But now formulas are more complicated.

$$x, y \neq 0 \rightarrow \frac{1}{((x/y)/x)} = y$$

becomes

$$\text{Div}(x, y, u) \land \text{Div}(u, x, v) \land \text{Div}(1, v, w) \land x, y \neq 0 \rightarrow w = y$$
Can we represent the concept of *square roots* with a function $\sqrt{} : \text{real} \Rightarrow \text{real}$?

- All positive real numbers have *two* square roots, and yet a function maps points to *single* values.
- We can pick one of the values arbitrarily: say, the *positive (principal)* square root.
- Or we can have the function map every real to a *set* $\sqrt{} : \text{real} \Rightarrow \text{real set}$:
  $$\sqrt{x} \equiv \{y \mid x = y^2\}.$$ 
- But now we have two kinds of object: reals and sets of reals, and we cannot conveniently express:
  $$\sqrt{x}^2 = x$$
- Our representation of reals is no longer *self-contained*. 
Q) Can we represent the concept of square roots with a relation $Sqrt : real \times real \Rightarrow bool$?

A) Yes. E.g. $Sqrt(x, y) \equiv x = y^2$.

Again drawback of formulas being more complicated
Functions, Predicates and Sets

We can translate back and forth. But too much translation makes a formalisation hard to use!

Any function $f : \alpha \to \beta$ can be represented as a relation $R : \alpha \times \beta \to bool$ or a set $S : (\alpha \times \beta) set$ by defining:

$$R(x, y) \equiv f(x) = y$$
$$S \equiv \{(x, y) \mid f(x) = y\}.$$

Any predicate $P$ can be represented by a function $f$ or a set $S$ by defining:

$$f(x) \equiv \begin{cases} True & : \ P(x) \\ False & : \ otherwise \end{cases}$$
$$S \equiv \{x \mid P(x)\}.$$

Any set $S$ can be represented by a function $f$ or a predicate $P$ by defining:

$$f(x) \equiv \begin{cases} True & : \ x \in S \\ False & : \ otherwise \end{cases}$$
$$P(x) \equiv x \in S.$$
In pure (without axioms) FOL, we cannot directly represent the statement:

there is a function that is larger on all arguments than the log function.

To formalise it, we would need to quantify over functions:

$$\exists f. \forall x. f(x) > \log x.$$

Likewise we cannot quantify over predicates.

**Solutions in FOL:**

- Represent all functions and predicates by sets, and quantify over these. This is the approach of first-order set theories such as ZF.
- Introduce sorts for predicates and functions. Not so elegant now having 2 kinds of each.
Summary

- This time:
  - Issues involved in representing mathematical theories
  - Axioms vs. Definitions
  - Functions vs. Predicates
  - Introduction to Higher-Order Logic
  - Reading: Bundy, Chapter 4 (contains further discussion of issues in representation, e.g. variadic functions).

- On the course web-page: some more exercises, asking you to “prove” False from the axioms of Naive Set Theory.