Automated Reasoning

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Natural Deduction in Propositional Logic (II)[1]

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[1] With contributions by Paul Jackson

Recap

- Last time we introduced **natural deduction**.
- We looked at the introduction rule *conjI*

$$\frac{P \quad Q}{P \wedge Q}\ conjI$$

- Now for the other rules of our formal deductive system ....

Conjunction Elimination

- **Elimination rules** work in the opposite direction to introduction rules

$$\frac{P \wedge Q}{P}\ conjunct1 \qquad \frac{P \wedge Q}{Q}\ conjunct2 \quad .$$

- An alternative conjunction elimination rule is:

$$\frac{P \wedge Q \qquad \begin{array}{c}[P] \quad [Q] \\ \vdots \\ R\end{array}}{R}\ conjE$$

P and Q are assumptions in a proof of R.

This can be useful for mechanized proofs.

Disjunction Elimination

Likewise, there are introduction and elimination rules for disjunction:

$$\frac{P}{P \vee Q}\ \mathit{disjI}1 \qquad \frac{Q}{P \vee Q}\ \mathit{disjI}2$$

$$\frac{P \vee Q \quad \begin{matrix}[P] \\ \vdots \\ R\end{matrix} \quad \begin{matrix}[Q] \\ \vdots \\ R\end{matrix}}{R}\ \mathit{disjE}$$

To show R given $P \vee Q$, it suffices to show R given P and R given Q.

The $[\phi]$ notation indicates that the rule discharges assumption ϕ.

Natural Deduction as Calculus of Derivations

Introduce sequent notation for derivations

$$\Gamma \vdash \phi$$

Conclusion ϕ is derivable from assumptions Γ

Elimination rules can then be formulated using these sequents. For instance:

$$\dfrac{P \vee Q \qquad \begin{array}{c}[P]\\ \vdots \\ R\end{array} \qquad \begin{array}{c}[Q]\\ \vdots \\ R\end{array}}{R}\ disjE$$

can be re-expressed as

$$\dfrac{\Gamma \vdash P \vee Q \qquad \Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma \vdash R}\ disjE$$

Disjunction Rules in Action

Prove that $B \vee A$ is true given that the assumption $A \vee B$ holds.

$$\dfrac{A \vee B \qquad \dfrac{[A]_1}{B \vee A}\,\mathit{disjI2} \qquad \dfrac{[B]_1}{B \vee A}\,\mathit{disjI1}}{B \vee A}\,\mathit{disjE}_1$$

Rules:

$$\dfrac{P}{P \vee Q}\,\mathit{disjI1}$$

$$\dfrac{Q}{P \vee Q}\,\mathit{disjI2}$$

$$\dfrac{P \vee Q \qquad \begin{array}{c}[P]\\ \vdots \\ R\end{array} \qquad \begin{array}{c}[Q]\\ \vdots \\ R\end{array}}{R}\,\mathit{disjE}$$

Subscript 1 indicates which rule discharges which hypotheses

Rules for Implication

$$\dfrac{\begin{matrix}[P] \\ \vdots \\ Q\end{matrix}}{P \longrightarrow Q}\ impl$$

impl Forward: If on the assumption that P is true, Q can be shown to hold, then we can conclude $P \longrightarrow Q$.
impl Backward: To prove $P \longrightarrow Q$, assume P is true and prove that Q follows.

$$\dfrac{P \longrightarrow Q \quad P}{Q}\ mp$$

The familiar **modus ponens** (mp) rule.

$$\dfrac{P \longrightarrow Q \quad P \quad \begin{matrix}[Q] \\ \vdots \\ R\end{matrix}}{R}\ impE$$

Another possible implication rule is this one. Note: this is not necessarily a standard ND rule but may be useful in mechanized proofs.

In general derivation assumptions may occur multiple times, and only a subset of the occurrences need be discharged.

Other Rules

Negation:

$$\frac{\begin{array}{c}[P]\\ \vdots\\ False\end{array}}{\neg P}\ notI \qquad \frac{\neg P \quad P}{R}\ notE\ .$$

If and only if ($\longleftrightarrow$):

$$\frac{\begin{array}{c}[Q]\\ \vdots\\ P\end{array} \quad \begin{array}{c}[P]\\ \vdots\\ Q\end{array}}{P \longleftrightarrow Q}\ iffI \qquad \frac{Q \longleftrightarrow P \quad Q}{P}\ iffD1 \qquad \frac{P \longleftrightarrow Q \quad Q}{P}\ iffD2$$

Proof

Recall the logic problems from lecture 2: we can now prove

$$(Sunny \vee Rainy) \wedge \neg Sunny \longrightarrow Rainy$$

as follows

$$\dfrac{\dfrac{\dfrac{[(S \vee R) \wedge \neg S]_1}{(S \vee R)}\,{\scriptstyle conjunct1} \qquad \dfrac{[S]_2 \qquad \dfrac{[(S \vee R) \wedge \neg S]_1}{\neg S}\,{\scriptstyle conjunct2}}{R}\,{\scriptstyle notE} \qquad [R]_2}{R}\,{\scriptstyle disjE_2}}{(S \vee R) \wedge \neg S \longrightarrow R}\,{\scriptstyle implI_1}$$

Problematic lemma

- Can every valid statement be proved using only the inference rules we have encountered so far?
- Consider Peirce's Law: $((A \longrightarrow B) \longrightarrow A) \longrightarrow A$.
- Our inference rules cannot prove this!
- **Classical logic**: We can prove it with the law of excluded middle: $\overline{A \vee \neg A}$
- **Intuitionistic logic**: It cannot generally be proved! Evidence must be given for every statement, so we would need either evidence of A or a refutation of A to assert $A \vee \neg A$.

Classical Rules in Isabelle/HOL

- The logic we will be using in Isabelle (HOL) is a classical logic and adds the rule:

$$\frac{}{P = True \lor P = False}\ True\_or\_False$$

- From this, we can derive the rules

$$\frac{}{\neg P \lor P}\ excluded\_middle \qquad \frac{\begin{array}{c}[\neg P]\\ \vdots \\ False\end{array}}{P}\ ccontr$$

Summary of Rules

$\wedge$	$\dfrac{P \quad Q}{P \wedge Q}\ \mathit{conjI}$	$\dfrac{P \wedge Q}{P}\ \mathit{conjunct1}$	$\dfrac{P \wedge Q}{Q}\ \mathit{conjunct2}$	$\dfrac{P \wedge Q \quad \begin{array}{c} P \quad Q \\ \vdots \\ R \end{array}}{R}\ \mathit{conjE}$
$\vee$	$\dfrac{P}{P \vee Q}\ \mathit{disjI1}$	$\dfrac{Q}{P \vee Q}\ \mathit{disjI2}$	$\dfrac{P \vee Q \quad \begin{array}{c} P \\ \vdots \\ R \end{array} \quad \begin{array}{c} Q \\ \vdots \\ R \end{array}}{R}\ \mathit{disjE}$	
$\rightarrow$	$\dfrac{\begin{array}{c} P \\ \vdots \\ Q \end{array}}{P \longrightarrow Q}\ \mathit{impI}$	$\dfrac{P \longrightarrow Q \quad P \quad \begin{array}{c} [Q] \\ \vdots \\ R \end{array}}{R}\ \mathit{impE}$		
$\leftrightarrow$	$\dfrac{\begin{array}{c} Q \\ \vdots \\ P \end{array} \quad \begin{array}{c} P \\ \vdots \\ Q \end{array}}{P \longleftrightarrow Q}\ \mathit{iffI}$	$\dfrac{Q \longleftrightarrow P \quad Q}{P}\ \mathit{iffD1}$	$\dfrac{P \longleftrightarrow Q \quad Q}{P}\ \mathit{iffD2}$	
$\neg$	$\dfrac{\begin{array}{c} [P] \\ \vdots \\ \mathit{False} \end{array}}{\neg P}\ \mathit{notI}$	$\dfrac{\neg P \quad P}{R}\ \mathit{notE}$	$\dfrac{\begin{array}{c} \neg P \\ \vdots \\ \mathit{False} \end{array}}{P}\ \mathit{ccontr}$	$\dfrac{}{A \vee \neg A}\ \mathit{excluded\_middle}$

Summary

- Natural deduction in propositional logic.
- Concept of a sequent calculus

Next time: Propositional reasoning in Isabelle